Using Tensor Diagrams to Represent and Solve Geometric Problems

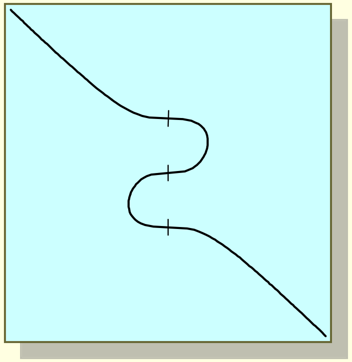
Introduction

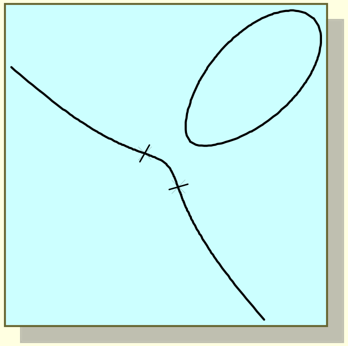
A Geometry Problem
Cubic Curves

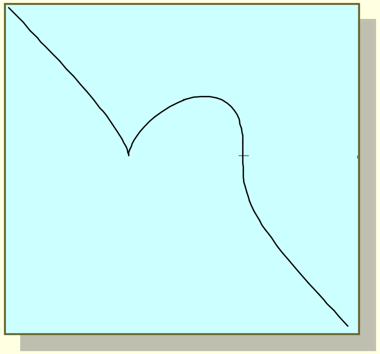
$$f(X,Y) = AX^3 + 3BX^2Y + 3CXY^2 + DY^3$$
$$+3EX^2 + 6FXY + 3GY^2$$
$$+3HX + 3JY$$
$$+K = 0$$

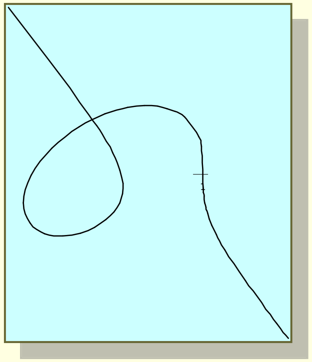
Possible Cubic Curve Shapes

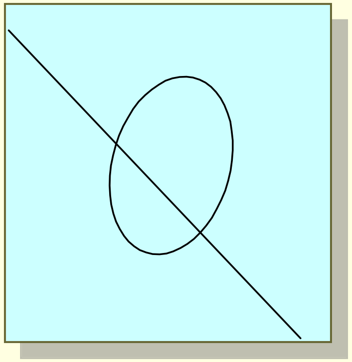
$$f(X,Y) = AX^3 + 3BX^2Y + 3CXY^2 + DY^3$$
$$+ 3EX^2 + 6FXY + 3GY^2$$
$$+ 3HX + 3JY$$
$$+ K = 0$$







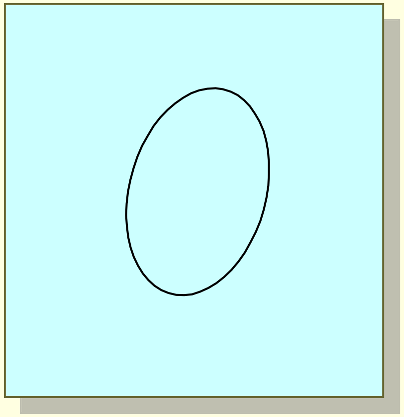


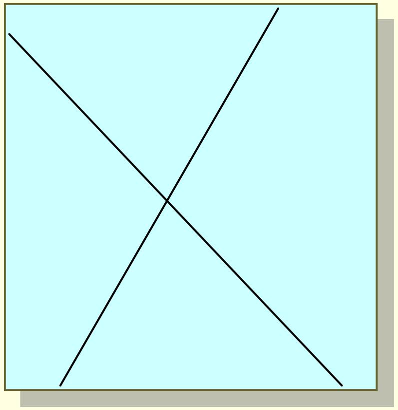


$$\mathbf{D}(A, B, C, \ldots, J, K)$$

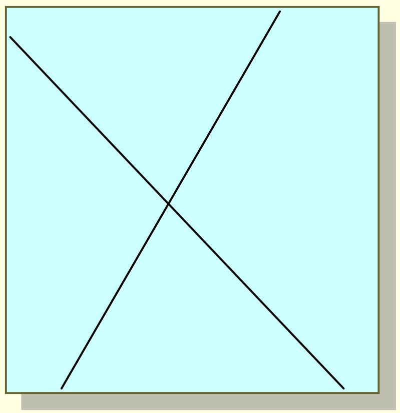
Quadratic Curves

$$f(X,Y) = AX^2 + 2BXY + CY^2 + 2DX + 2EY + F = 0$$





Discriminant of Quadratic



$$\mathbf{D}\left(A,B,C,D,E\right)=0$$

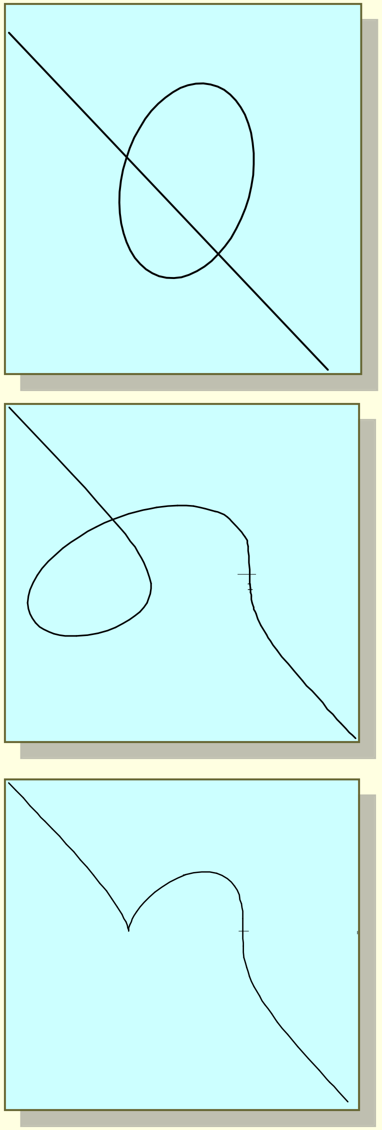
$$ACF+2BED-D^2C-E^2A-B^2F=0$$

D is degree 3 in A…E

D has 5 terms

$$\det\begin{bmatrix}A & B & D\\ B & C & E\\ D & E & F\end{bmatrix}=0$$

Discriminant of Cubic



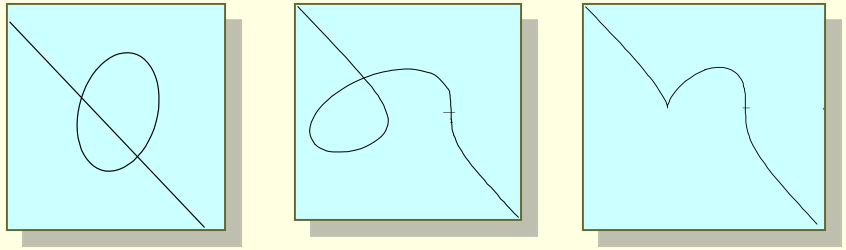
$$\mathbf{D}\left(A,B,C,D,E,F,G,H,J,K\right) = 0$$

G. Salmon (1879):

D is degree 12 in *A*…*K*

D has over 10,000 terms

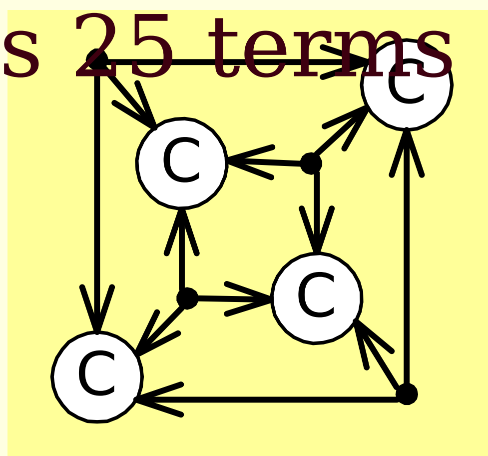
Discriminant of Cubic



$$\mathbf{D} = 64S^3 + T^2$$

S: degree 4 in $A \ldots K$

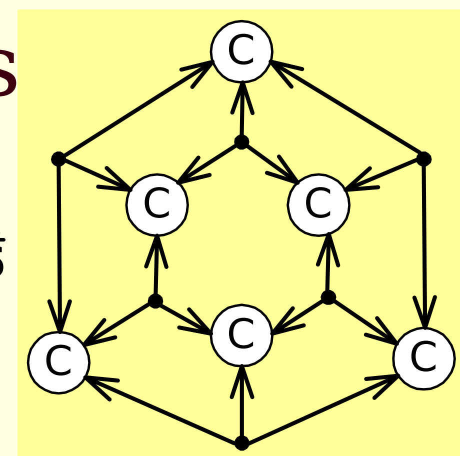
has 25 terms



$$S = -\frac{1}{24}$$

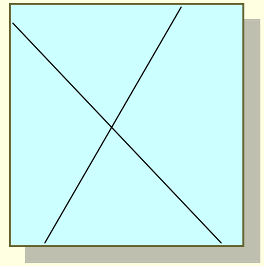
T: degree 6 in $A \ldots K$

has



$$T = -\frac{1}{6}$$

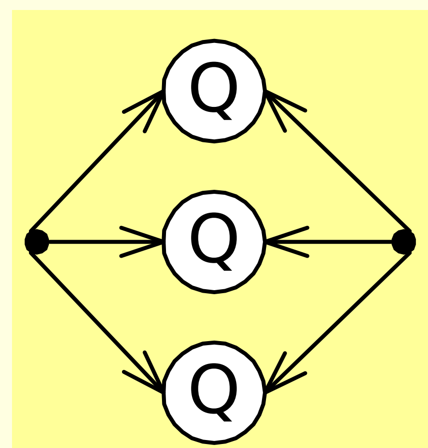
Discriminant of Quadratic



$$\Delta(A,B,C,D,E)=0$$

$$ACF+2BED-D^2C-E^2A-B^2F=0$$

$$\det\begin{bmatrix}A & B & D\\ B & C & E\\ D & E & F\end{bmatrix}=0$$



$=0$

Tensor Diagrams

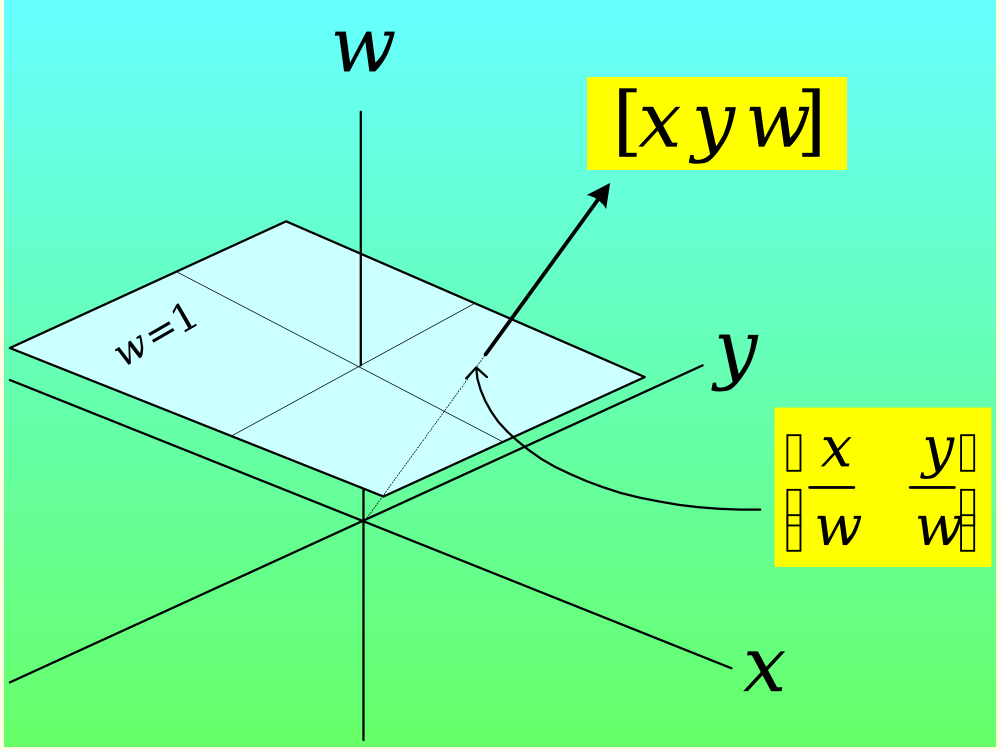
- Express complicated polynomials visually
- Aid in manipulation of polynomials
- Derive existing results easily
- Derive new results (?)

Homogeneous Geometry

$$P = [X \quad Y]$$

$$P = [x \quad y \quad w]$$

$$aP = [ax \quad ay \quad aw]$$



The Homogeneous Universes

1D (Polynomials)

$$f(X) = AX^2 + BX + C$$

$$f(x,w) = Ax^2 + Bxw + Cw^2$$

2D (Curves)

$$f(X,Y) = DX^2 + EY + F$$

$$f(x,y,w) = Dx^2 + Eyw + Fw^2$$

3D (Surfaces)

$$f(X,Y,Z) = GX^2 + HY + JZ$$

$$f(x,y,z,w) = Gx^2 + Hyw + Jzw$$

The Homogeneous Universes

	Euclidean	Projective
Polynomials	1D: $[X]$	1DH: $[x\ w]$
Curves	2D: $[X\ Y]$	2DH: $[x\ y\ w]$
Surfaces	3D: $[X\ Y\ Z]$	3DH: $[x\ y\ z\ w]$

The Matrix of Knowledge

	1DH	2DH	3DH
Linear			
Quadratic			
Pairs of quadratics			
Cubic			
Quadratic and Cubic			
Pair of Cubics			
Quartic			

Tensor Diagrams

- A Work in Progress
  - Some simple results not complete
  - Lot of stuff is still rough around the edges
  - Tutorial notes are obsolete
- Want to show what I’ve figured out so far
- Enlist others in finding more results

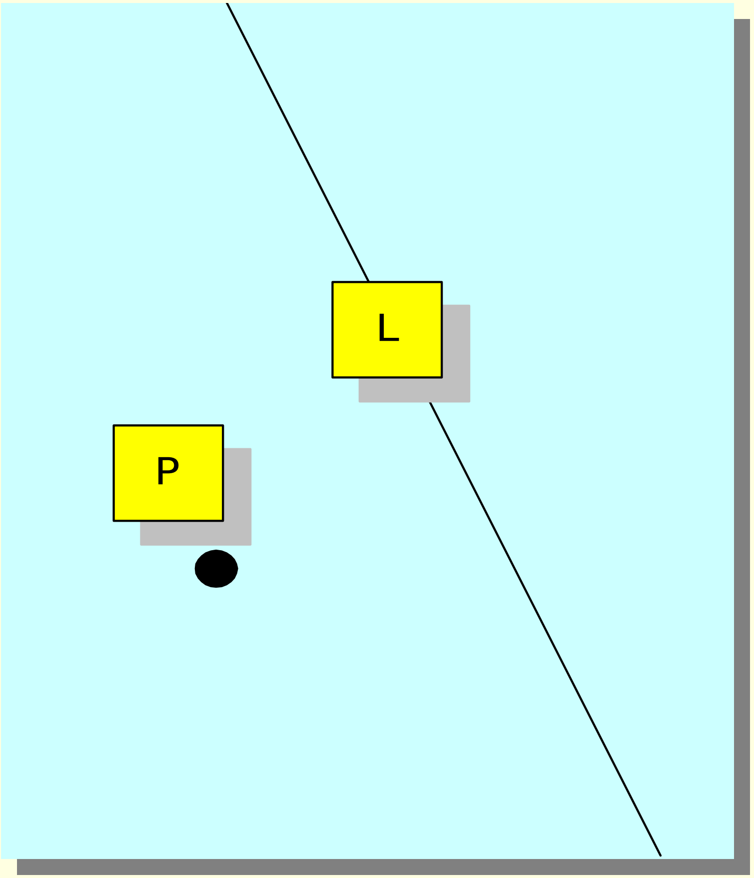
Basic Homogeneous Geometry

The Problems

2DH Points and Lines

$$\mathbf{P} = \begin{bmatrix} x & y & w \end{bmatrix}$$

$$\mathbf{L} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

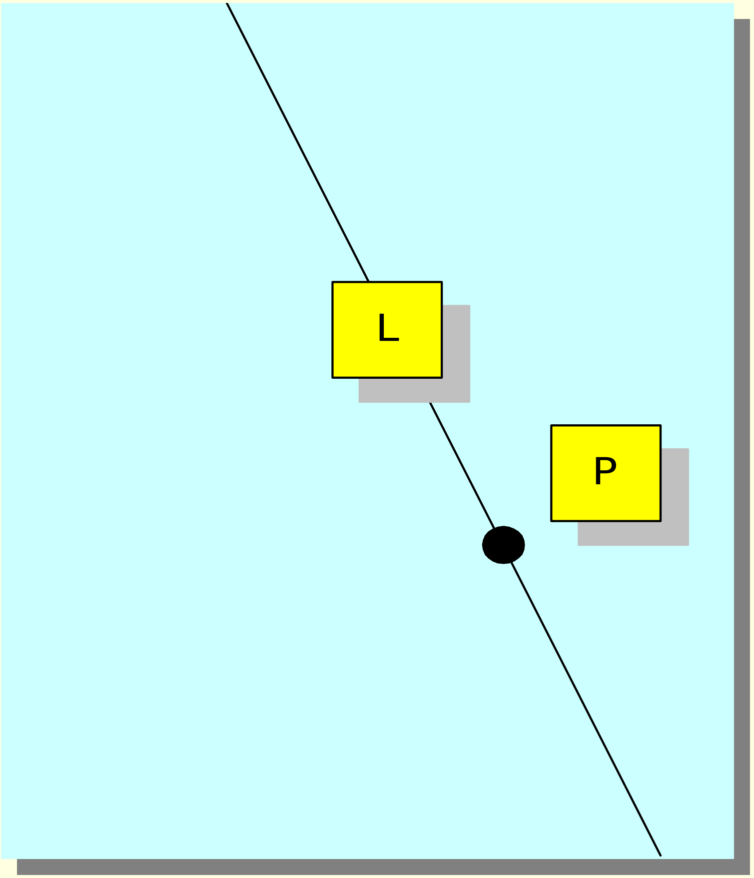


2DH Point on a Line

$$ax + by + cw = 0$$

$$\begin{bmatrix} x & y & w \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$$\mathbf{P} \cdot \mathbf{L} = 0$$



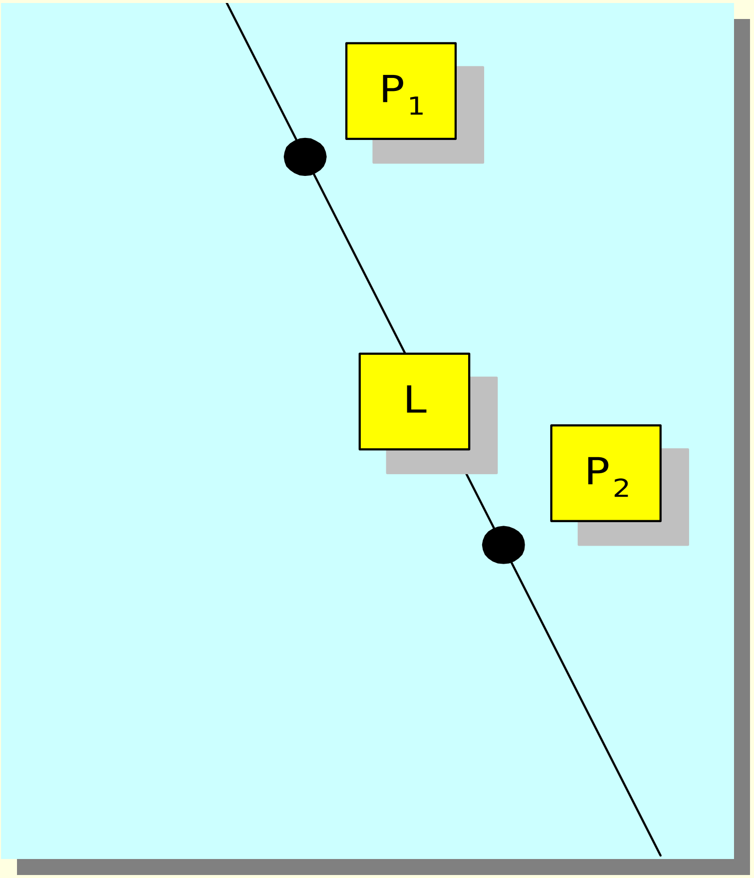
2DH Two Points Make A Line

$$\mathbf{P}_1 \times \mathbf{P}_2 = \mathbf{L}$$

$$\begin{bmatrix} x_1 & y_1 & z_1 \end{bmatrix} \times \begin{bmatrix} x_2 & y_2 & z_2 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

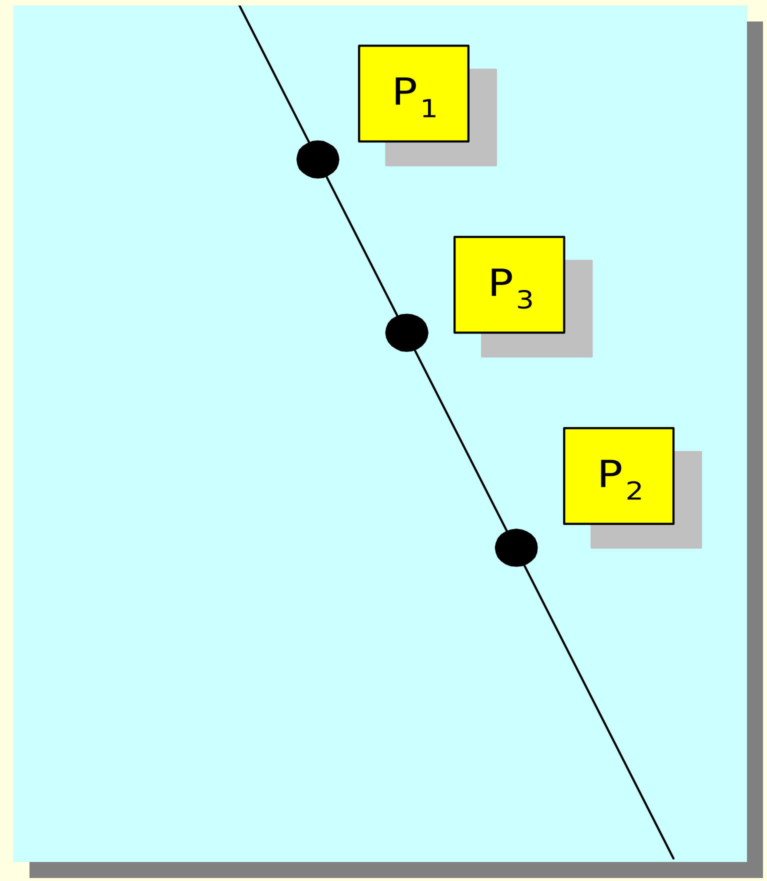
$$a = \det \begin{bmatrix} y_1 & w_1 \\ y_2 & w_2 \end{bmatrix}$$

$$b = \det \begin{bmatrix} w_1 & x_1 \\ w_2 & x_2 \end{bmatrix} \quad c = \det \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$$



2DH Three Collinear Points

$$\mathbf{P}_1 \times \mathbf{P}_2 \cdot \mathbf{P}_3 = 0$$



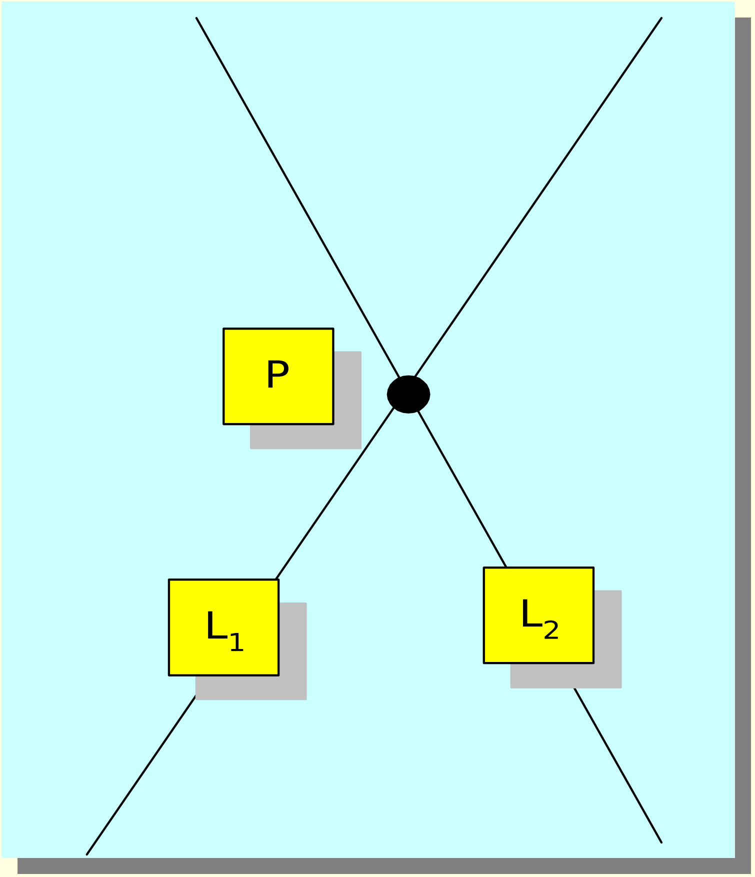
2DH Two Lines Make A Point

$$\mathbf{L}_1 \times \mathbf{L}_2 = \mathbf{P}$$

$$\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} \times \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} x & y & w \end{bmatrix}$$

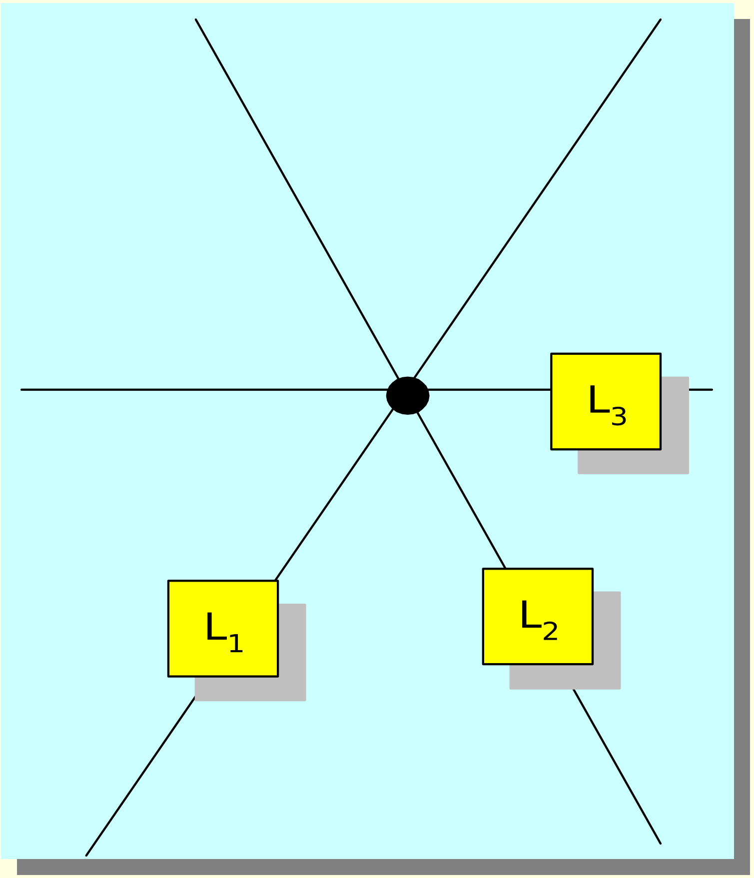
$$x = \det \begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix} \qquad y = \det \begin{bmatrix} c_1 & c_2 \\ a_1 & a_2 \end{bmatrix}$$

$$z = \det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$$



2DH Three CoPointar Lines

$$\mathbf{L}_1 \times \mathbf{L}_2 \cdot \mathbf{L}_3 = 0$$



2DH Transforming Points

$$\mathbf{PT} = \hat{\mathbf{P}}$$

$$\begin{bmatrix} x & y & w \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} = \begin{bmatrix} \hat{x} & \hat{y} & \hat{w} \end{bmatrix}$$

2DH Transforming Lines

$$\mathbf{P}\cdot\mathbf{L}=0 \Leftrightarrow \hat{\mathbf{P}}\cdot\hat{\mathbf{L}}=0$$

$$\begin{aligned}\mathbf{P}\cdot\mathbf{L} &= \mathbf{P}\left(\mathbf{T}\mathbf{T}^{-1}\right)\mathbf{L}\\ &= \left(\mathbf{P}\mathbf{T}\right)\left(\mathbf{T}^{-1}\mathbf{L}\right)\\ &= \hat{\mathbf{P}}\left(\mathbf{T}^{-1}\mathbf{L}\right)\end{aligned}$$

$$\boxed{\mathbf{T}^{-1}\mathbf{L}=\hat{\mathbf{L}}}$$

2DH Matrix Adjoint

$$\mathbf{T} = \begin{bmatrix} \cdots & R_1 & \cdots \\ \cdots & R_2 & \cdots \\ \cdots & R_3 & \cdots \end{bmatrix} \qquad \mathbf{T}^* = \begin{bmatrix} \vdots & \vdots & \vdots \\ R_2 \times R_3 & R_3 \times R_1 & R_1 \times R_2 \\ \vdots & \vdots & \vdots \end{bmatrix}$$

$$\mathbf{T}\mathbf{T}^* = \left(\det \mathbf{T}\right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{T}^* = \left(\det \mathbf{T}\right) \mathbf{T}^{-1}$$

2DH Transforming Points and Lines

$$\mathbf{P}\mathbf{T} = \hat{\mathbf{P}} \qquad \begin{bmatrix} x & y & w \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} = \begin{bmatrix} \hat{x} & \hat{y} & \hat{w} \end{bmatrix}$$

$$\mathbf{T}^{*}\mathbf{L} = \hat{\mathbf{L}} \qquad \begin{bmatrix} T^{*}_{11} & T^{*}_{12} & T^{*}_{13} \\ T^{*}_{21} & T^{*}_{22} & T^{*}_{23} \\ T^{*}_{31} & T^{*}_{32} & T^{*}_{33} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \hat{a} \\ \hat{b} \\ \hat{c} \end{bmatrix}$$

2DH Point on Quadratic Curve

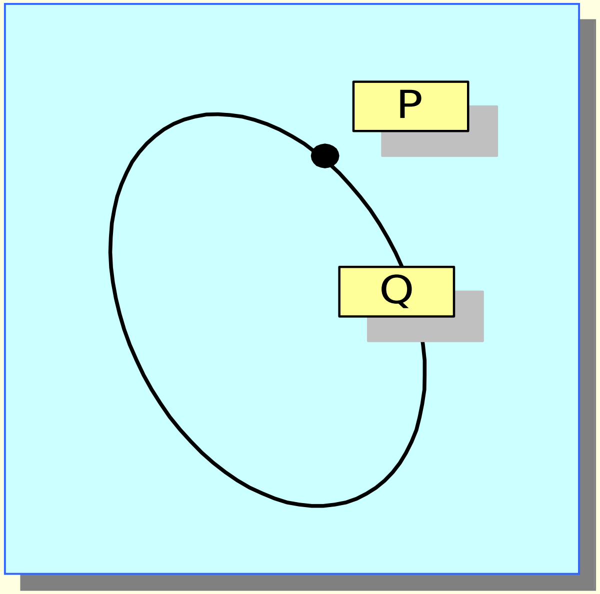
$$Ax^2 + 2Bxy + 2Cxw$$

$$+ Dy^2 + 2Eyw$$

$$+ Fw^2 = 0$$

$$\begin{bmatrix} x & y & w \end{bmatrix} \begin{bmatrix} A & B & C \\ B & D & E \\ C & E & F \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = 0$$

$$\mathbf{P} \cdot \mathbf{Q} \cdot \mathbf{P}^T = 0$$



2DH Transforming a Quadratic

$$\mathbf{P}\mathbf{Q}\mathbf{P}^T = 0 \Leftrightarrow \hat{\mathbf{P}}\hat{\mathbf{Q}}\hat{\mathbf{P}}^T = 0$$

$$\mathbf{P}\mathbf{Q}\mathbf{P}^T = d^2\, \mathbf{P}\left(\mathbf{T}\mathbf{T}^*\right)\mathbf{Q}\left(\mathbf{T}\mathbf{T}^*\right)^T \mathbf{P}^T$$

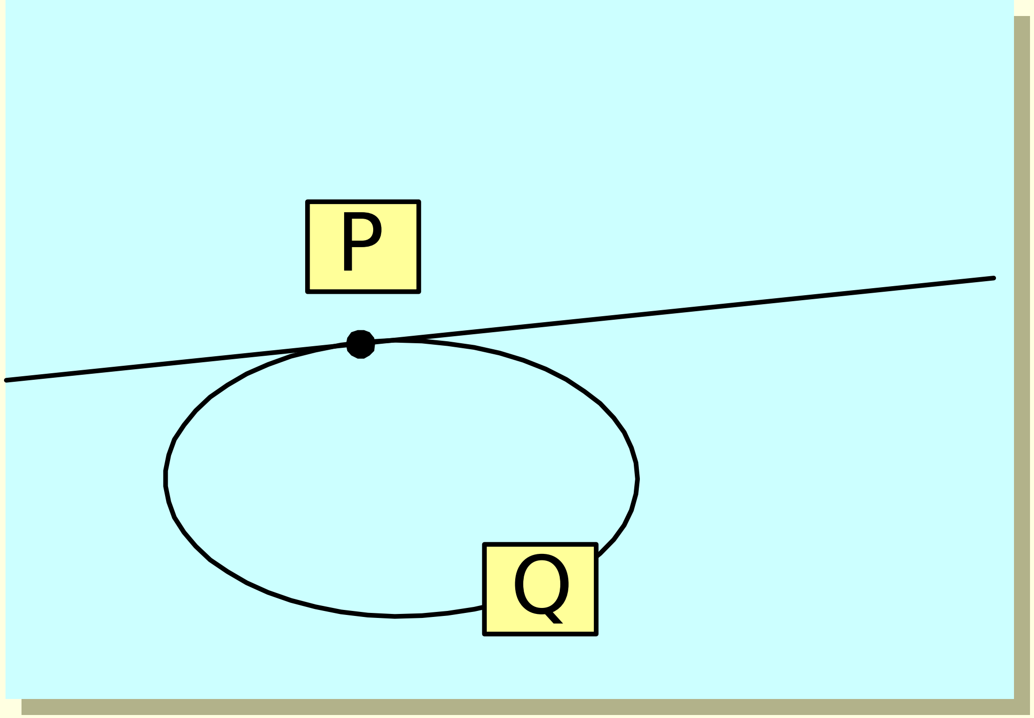
$$= d^2 \left(\mathbf{P}\mathbf{T}\right)\left(\mathbf{T}^*\mathbf{Q}\mathbf{T}^{*T}\right)\left(\mathbf{P}\mathbf{T}\right)^T$$

$$= d^2\, \hat{\mathbf{P}}\left(\mathbf{T}^*\mathbf{Q}\mathbf{T}^{*T}\right)\hat{\mathbf{P}}^T$$

$$\boxed{\mathbf{T}^*\mathbf{Q}\mathbf{T}^{*T} = \hat{\mathbf{Q}}}$$

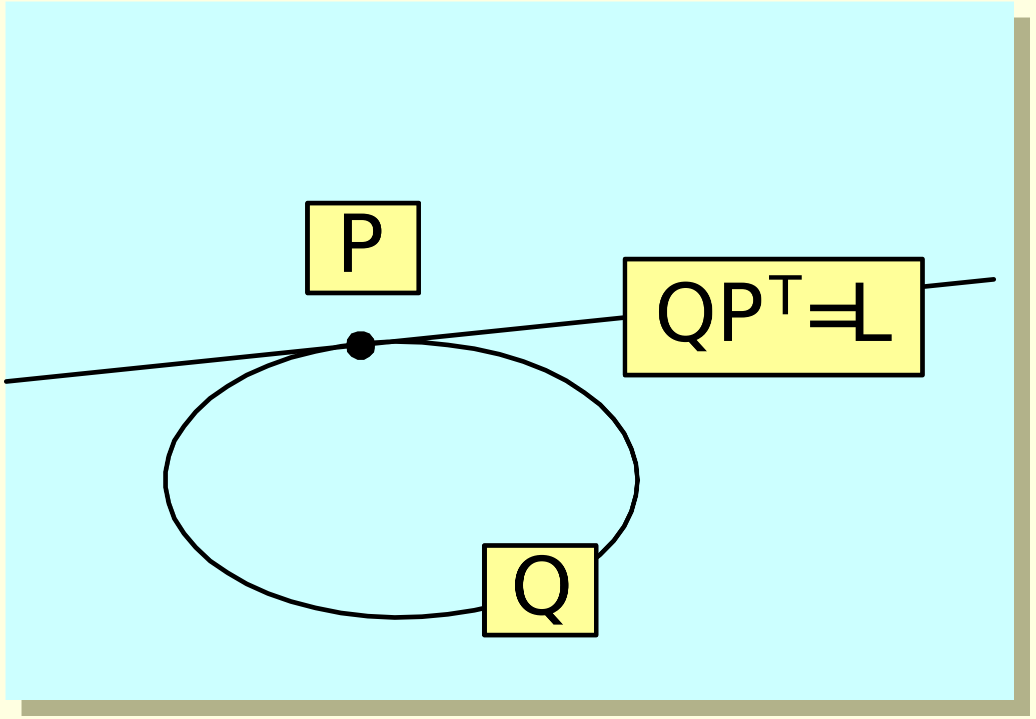
2DH Tangent at a Point

$$
\begin{aligned}
0 &= \mathbf{PQP}^T \\
&= \mathbf{P} \times \left(\mathbf{QP}^T \right) \\
&= \mathbf{P} \times \mathbf{L}
\end{aligned}
$$



2DH Tangent at a Point

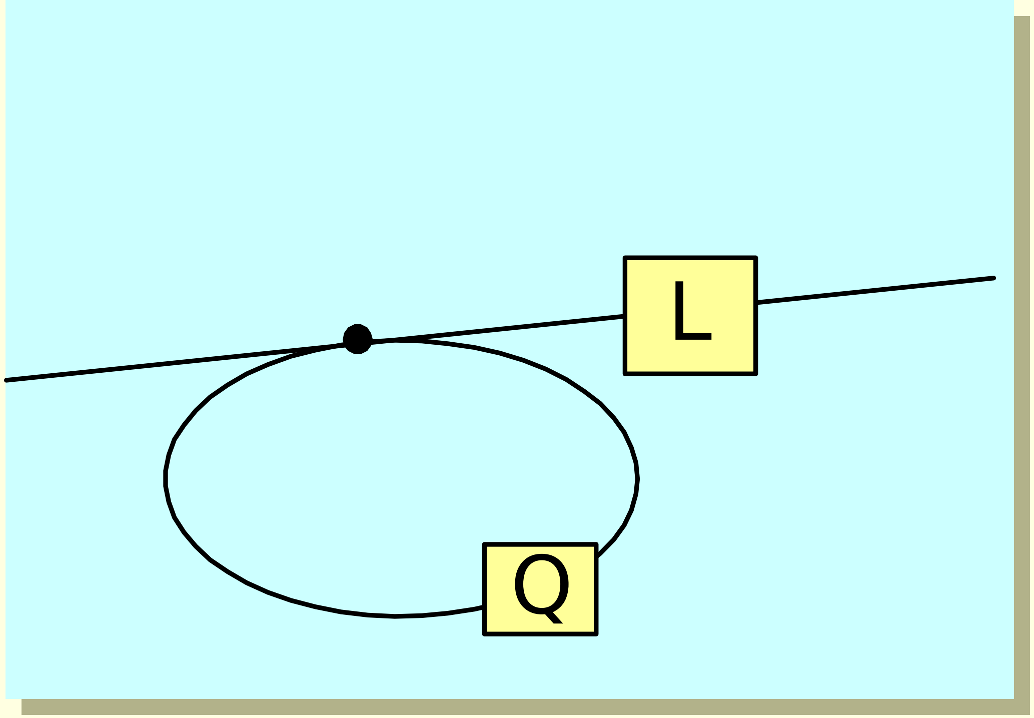
$$
\begin{aligned}
0 &= \mathbf{PQP}^T \\
&= \mathbf{P} \times \left(\mathbf{QP}^T\right) \\
&= \mathbf{P} \times \mathbf{L}
\end{aligned}
$$



2DH Line Tangent to Quadric

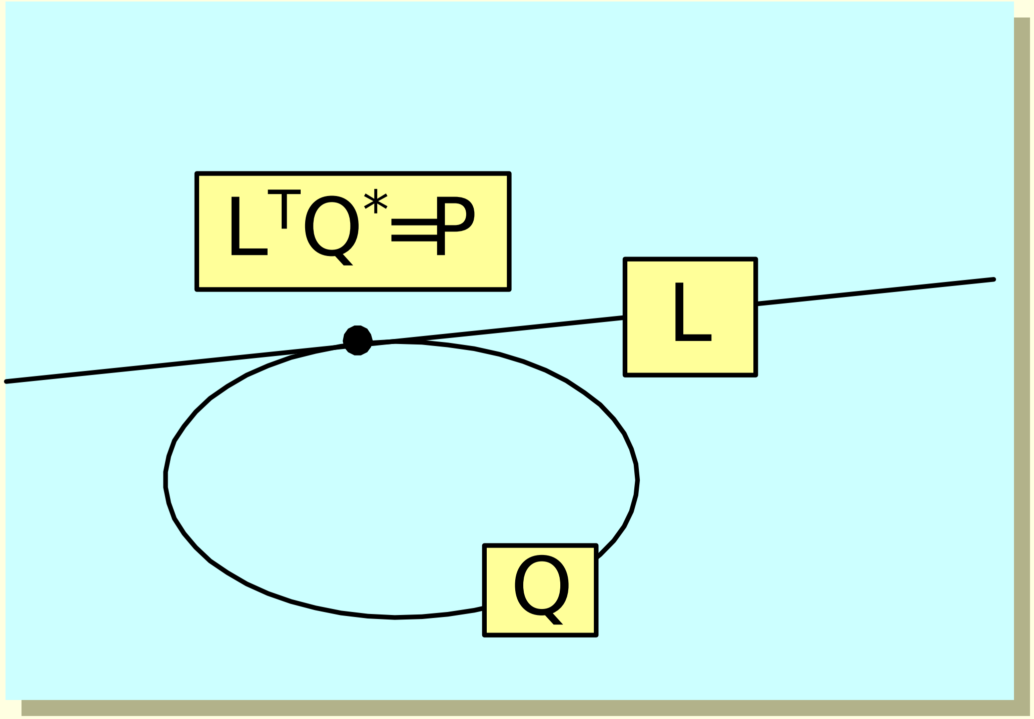
$$0 = \mathbf{L}^T \mathbf{Q}^* \mathbf{L}$$

$$= \left(\mathbf{L}^T \mathbf{Q}^* \right) \mathbf{L}$$



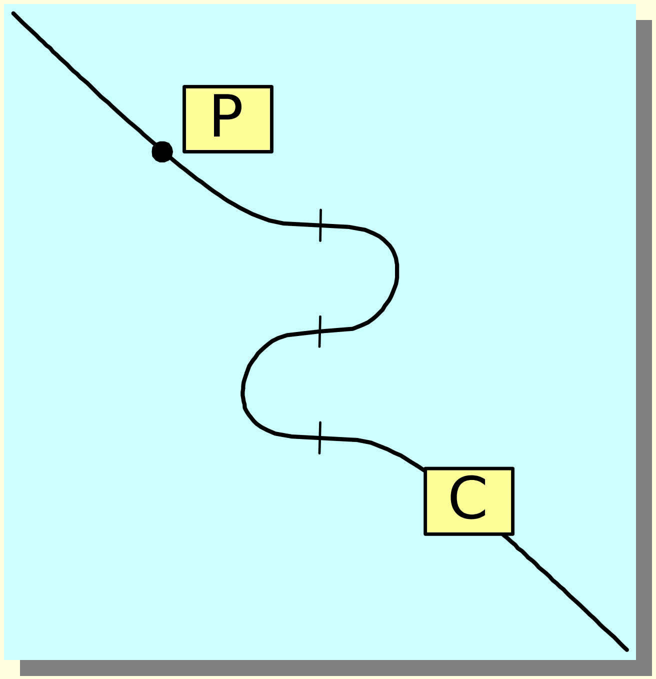
2DH Line Tangent to Quadric

$$0 = \mathbf{L}^T \mathbf{Q}^* \mathbf{L}$$
$$= \left(\mathbf{L}^T \mathbf{Q}^* \right) \mathbf{L}$$



2DH Point on Cubic Curve

$$Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3$$
$$+3Ex^2w + 6Fxyw + 3Gy^2w$$
$$+3Hxw^2 + 3Jyw^2$$
$$+Kw^3 = 0$$



2DH Cubic Curve

$$
\begin{aligned}
&Ax^2 + 3Bx^2y + 3Cxy^2 + Dy^3 \\
&+ 3Ex^2w + 6Fxyw + 3Gyw^2 \\
&+ 2Hxw^2 + 3Jyw^2 \\
&+ Kw^2 = 0
\end{aligned}
$$

$$
\left\{ \begin{bmatrix} x & y & w \end{bmatrix} \begin{bmatrix} \begin{bmatrix} A & B & E \\ B & C & F \\ E & F & H \end{bmatrix} & \begin{bmatrix} B & C & F \\ C & D & G \\ F & G & J \end{bmatrix} & \begin{bmatrix} E & F & H \\ F & G & J \\ H & J & K \end{bmatrix} \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} \right\} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = 0
$$

$$
\left\{ \mathbf{P} \mathbf{C} \mathbf{P}^T \right\} \mathbf{P}^T = 0
$$

2DH Curves of Various Orders

$$\mathrm{L} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \qquad \mathbf{Q} = \begin{bmatrix} A & B & C \\ B & D & E \\ C & E & F \end{bmatrix}$$

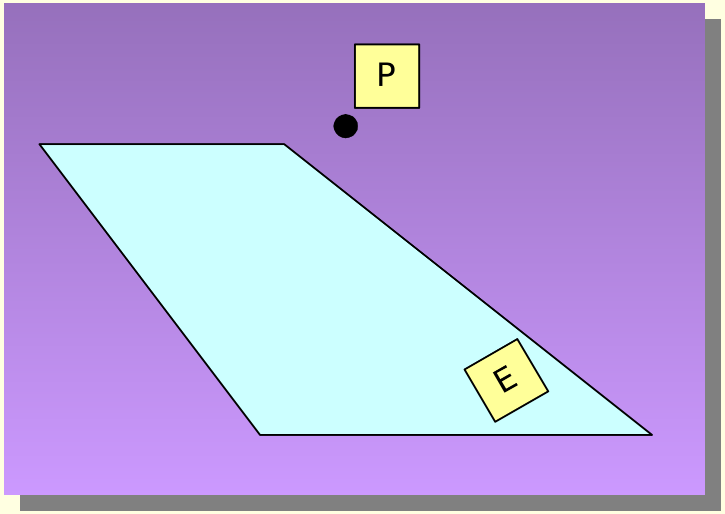
$$\mathbf{C} = \begin{bmatrix} \begin{bmatrix} A & B & E \\ B & C & F \\ E & F & H \end{bmatrix} & \begin{bmatrix} B & C & F \\ C & D & G \\ F & G & J \end{bmatrix} & \begin{bmatrix} E & F & H \\ F & G & J \\ H & J & K \end{bmatrix} \end{bmatrix}$$

Now 3D (Homogeneous)

3DH Points and Planes

$$P = \begin{bmatrix} x & y & z & w \end{bmatrix}$$

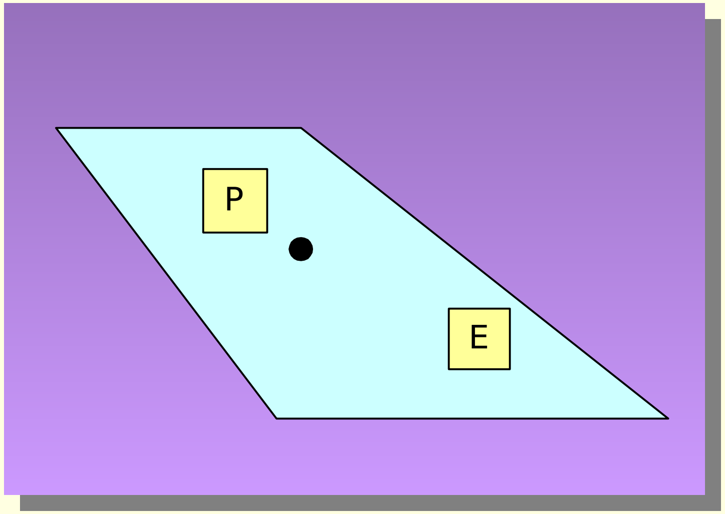
$$E = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$



3DH Point on Plane

$$\begin{bmatrix} x & y & z & w \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = 0$$

$$\mathbf{P} \times \mathbf{E} = 0$$



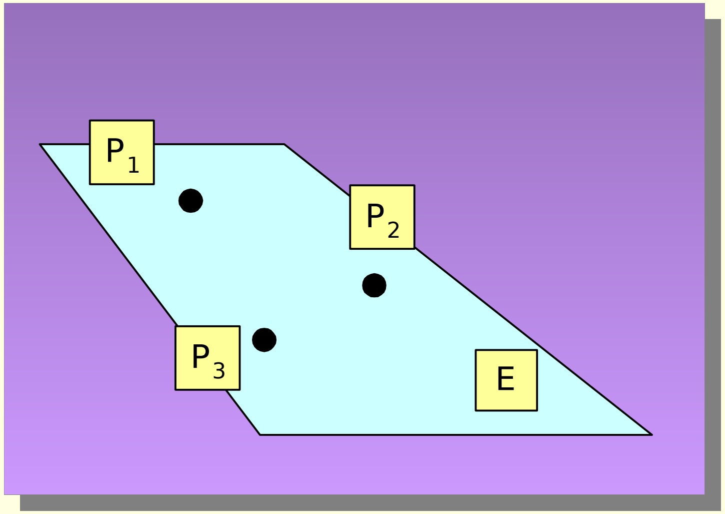
3DH Transformations

$$\mathbf{P}\mathbf{T} = \mathbf{P}'$$

$$\mathbf{T}^{*}\mathbf{E} = \mathbf{E}'$$

3DH Plane thru 3 Points

$$cross\left(\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3\right) = \mathbf{E}$$



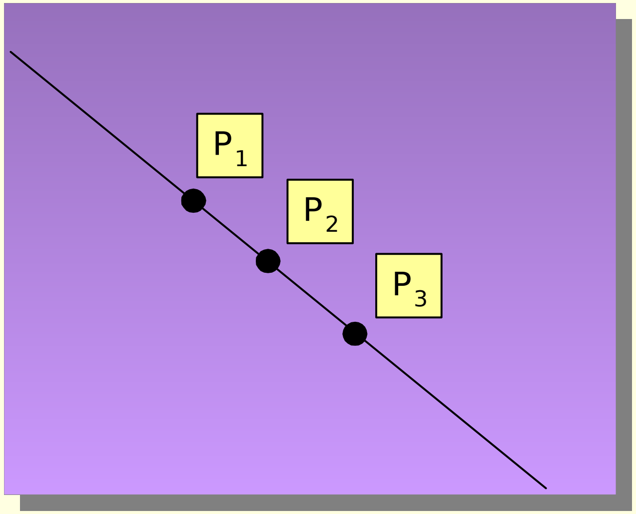
$$cross\begin{pmatrix} [x_1 & y_1 & z_1 & w_1], \\ [x_2 & y_2 & z_2 & w_2], \\ [x_3 & y_3 & z_3 & w_3] \end{pmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$a = \det\begin{bmatrix} y_1 & z_1 & w_1 \\ y_2 & z_2 & w_2 \\ y_3 & z_3 & w_3 \end{bmatrix}, \quad b = -\det\begin{bmatrix} x_1 & z_1 & w_1 \\ x_2 & z_2 & w_2 \\ x_3 & z_3 & w_3 \end{bmatrix}$$

$$c = \det\begin{bmatrix} y_1 & y_1 & w_1 \\ y_2 & y_2 & w_2 \\ y_3 & y_3 & w_3 \end{bmatrix}, \quad d = -\det\begin{bmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}$$

3DH Three Collinear points

$$cross\begin{pmatrix} [x_1 & y_1 & z_1 & w_1], \\ [x_2 & y_2 & z_2 & w_2], \\ [x_3 & y_3 & z_3 & w_3] \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



3DH Rewrite Equation

$$cross\begin{pmatrix} [x_1 \quad y_1 \quad z_1 \quad w_1], \\ [x_2 \quad y_2 \quad z_2 \quad w_2], \\ [x_3 \quad y_3 \quad z_3 \quad w_3] \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$0 = \det\begin{bmatrix} y_1 & z_1 & w_1 \\ y_2 & z_2 & w_2 \\ y_3 & z_3 & w_3 \end{bmatrix}$$

$$0 = y_3 \det\begin{bmatrix} z_1 & w_1 \\ z_2 & w_2 \end{bmatrix} - z_3 \det\begin{bmatrix} y_1 & w_1 \\ y_2 & w_2 \end{bmatrix} + w_3 \det\begin{bmatrix} y_1 & z_1 \\ y_2 & z_2 \end{bmatrix}$$

$$0 = \begin{bmatrix} 0 & \det\begin{bmatrix} z_1 & w_1 \\ z_2 & w_2 \end{bmatrix} & -\det\begin{bmatrix} y_1 & w_1 \\ y_2 & w_2 \end{bmatrix} & \det\begin{bmatrix} y_1 & z_1 \\ y_2 & z_2 \end{bmatrix} \end{bmatrix} \begin{bmatrix} x_3 \\ y_3 \\ z_3 \\ w_3 \end{bmatrix}$$

3DH Separate P_1P_2 from P_3

$$\begin{bmatrix} 0 & p & -q & r \\ -p & 0 & s & -t \\ q & -s & 0 & u \\ -r & t & -u & 0 \end{bmatrix} \begin{bmatrix} x_3 \\ y_3 \\ z_3 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$p = \det \begin{bmatrix} z_1 & w_1 \\ z_2 & w_2 \end{bmatrix} \quad q = \det \begin{bmatrix} y_1 & w_1 \\ y_2 & w_2 \end{bmatrix} \quad r = \det \begin{bmatrix} y_1 & z_1 \\ y_2 & z_2 \end{bmatrix}$$

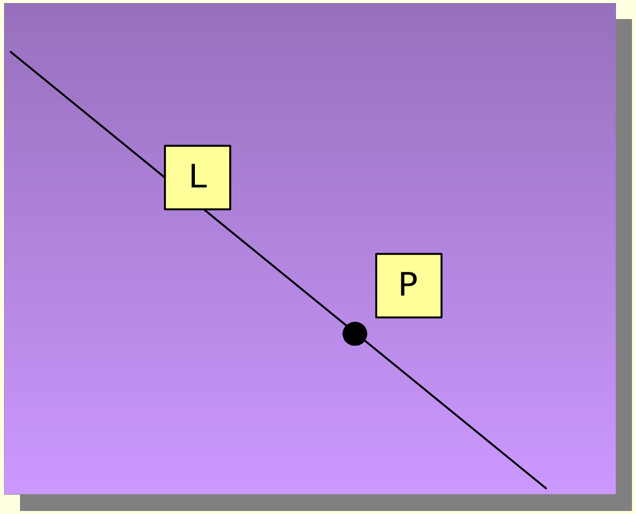
$$s = \det \begin{bmatrix} x_1 & w_1 \\ x_2 & w_2 \end{bmatrix} \quad t = \det \begin{bmatrix} x_1 & z_1 \\ x_2 & z_2 \end{bmatrix} \quad u = \det \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}$$

$$pu - qt + sr = 0$$

3DH Point on Line

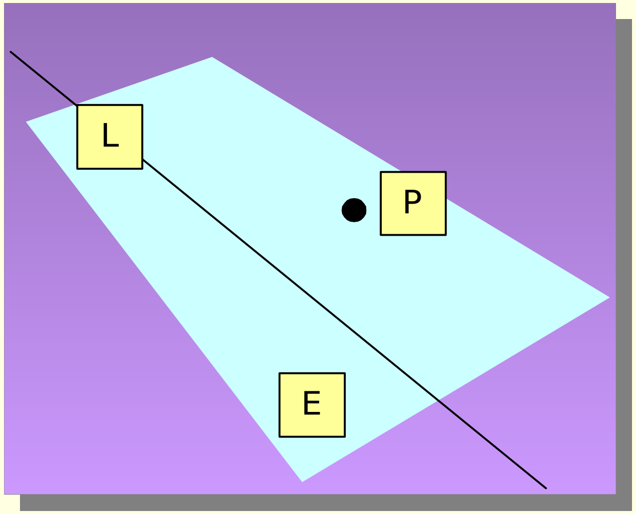
$$\begin{bmatrix} 0 & p & -q & r \\ -p & 0 & s & -t \\ q & -s & 0 & u \\ -r & t & -u & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{L}\mathbf{P}^T = \mathbf{0}$$



3DH Point not on Line = Plane

$$\begin{bmatrix} 0 & p & -q & r \\ -p & 0 & s & -t \\ q & -s & 0 & u \\ -r & t & -u & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$



$$\mathbf{L}\mathbf{P}^T = \mathbf{E}$$

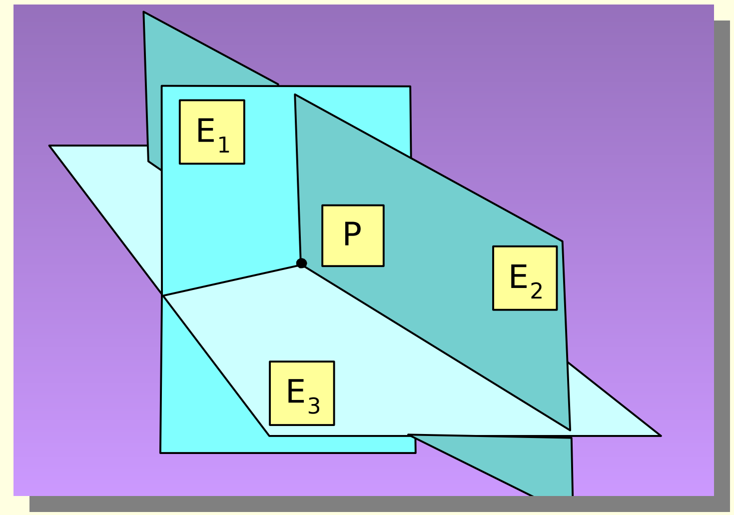
3DH Transforming a Line

$$\mathbf{L}\mathbf{P}^T = \mathbf{E} \Leftrightarrow \mathbf{L}'\mathbf{P}'^T = \mathbf{E}'$$

$$\boxed{\mathbf{T}^*\mathbf{L}\left(\mathbf{T}^*\right)^T = \mathbf{L}'}$$

3DH Point on 3 Planes

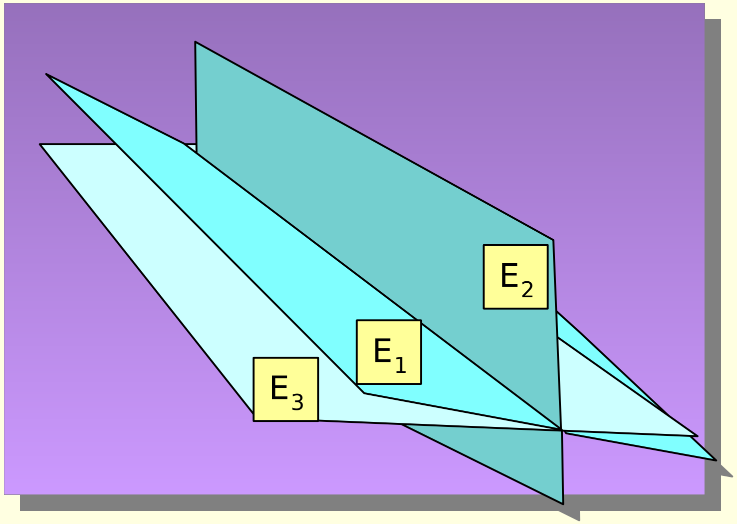
$$cross\left(\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\right) = \mathbf{P}$$



$$crs4\left(\begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{bmatrix}, \begin{bmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix}, \begin{bmatrix} a_3 \\ b_3 \\ c_3 \\ d_3 \end{bmatrix}\right) = \begin{bmatrix} x & y & z & w \end{bmatrix}$$

$$x = \det\begin{bmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{bmatrix}, \quad y = -\det\begin{bmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{bmatrix} \quad z = \det\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{bmatrix}, \quad w = -\det\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

3DH Three Collinear Planes



$$crs4\begin{pmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{bmatrix} & \begin{bmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix} & \begin{bmatrix} a_3 \\ b_3 \\ c_3 \\ d_3 \end{bmatrix} \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$$

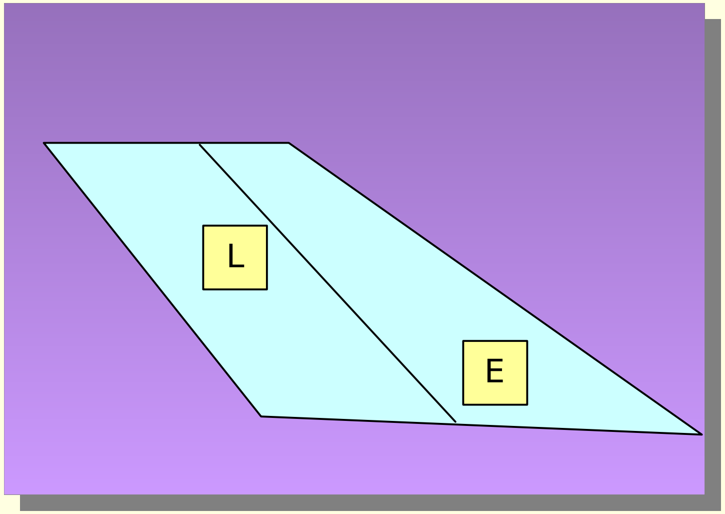
3DH Separate E_1E_2 from E_3

$$\begin{bmatrix} a_3 & b_3 & c_3 & d_3 \end{bmatrix} \begin{bmatrix} 0 & e & -f & g \\ -e & 0 & h & -j \\ f & -h & 0 & k \\ -g & j & -k & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$$

$$e = \det \begin{bmatrix} c_1 & c_2 \\ d_1 & d_2 \end{bmatrix} \quad f = \det \begin{bmatrix} b_1 & b_2 \\ d_1 & d_2 \end{bmatrix} \quad g = \det \begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}$$

$$h = \det \begin{bmatrix} a_1 & a_2 \\ d_1 & d_2 \end{bmatrix} \quad j = \det \begin{bmatrix} a_1 & a_2 \\ c_1 & c_2 \end{bmatrix} \quad k = \det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$$

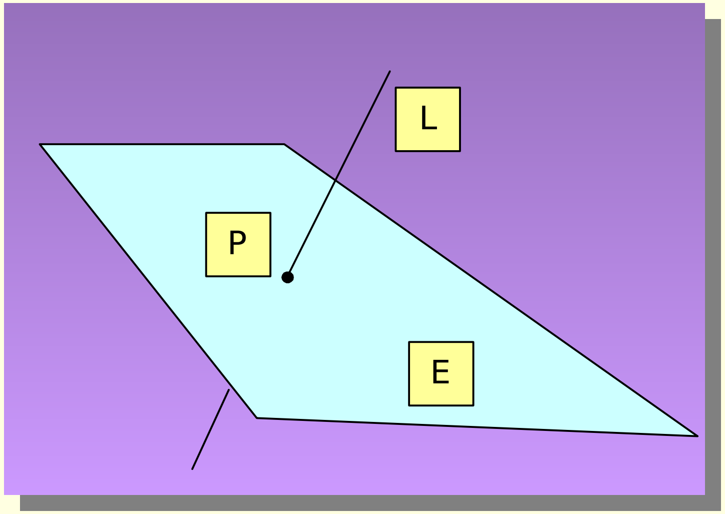
3DH Line embedded in Plane



$$\begin{bmatrix} a & b & c & d \end{bmatrix} \begin{bmatrix} 0 & e & -f & g \\ -e & 0 & h & -j \\ f & -h & 0 & k \\ -g & j & -k & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{E}^T \mathbf{K} = \mathbf{0}$$

3DH Line Not in Plane = Point



$$\begin{bmatrix} a & b & c & d \end{bmatrix} \begin{bmatrix} 0 & e & -f & g \\ -e & 0 & h & -j \\ f & -h & 0 & k \\ -g & j & -k & 0 \end{bmatrix} = \begin{bmatrix} x & y & z & w \end{bmatrix}$$

$$\mathbf{E}^T\mathbf{K} = \mathbf{P}$$

3DH Two Forms of Line

$$\begin{bmatrix} 0 & p & -q & r \\ -p & 0 & s & -t \\ q & -s & 0 & u \\ -r & t & -u & 0 \end{bmatrix} = \mathbf{L} \qquad \begin{bmatrix} 0 & e & -f & g \\ -e & 0 & h & -j \\ f & -h & 0 & k \\ -g & j & -k & 0 \end{bmatrix} = \mathbf{K}$$

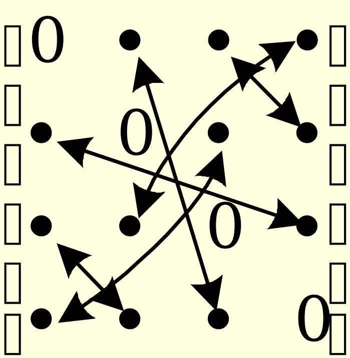
$$\mathbf{L}\mathbf{P}^T = \mathbf{E} \qquad \mathbf{E}^T\mathbf{K} = \mathbf{P}$$

3DH Converting Between Two Forms of Line

$$\mathbf{L}=\begin{bmatrix} 0 & p & -q & r \\ -p & 0 & s & -t \\ q & -s & 0 & u \\ -r & t & -u & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & e & -f & g \\ -e & 0 & h & -j \\ f & -h & 0 & k \\ -g & j & -k & 0 \end{bmatrix}=\mathbf{K}=\begin{bmatrix} 0 & -u & -t & -s \\ u & 0 & -r & -q \\ t & r & 0 & -p \\ s & q & p & 0 \end{bmatrix}$$

$$e=-u,\quad f=t,\quad g=-s$$
$$h=-r,\quad j=q,\quad k=-p$$



Two Problems

Rows vs. Columns

$$\begin{bmatrix} A & B & C \\ B & D & E \\ C & E & F \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{bmatrix} x & y & z & w \end{bmatrix} = \begin{bmatrix} a & b & c & d \end{bmatrix} \begin{bmatrix} 0 & e & -f & g \\ -e & 0 & h & -j \\ f & -h & 0 & k \\ -g & j & -k & 0 \end{bmatrix}$$

More than Two Indices

$$\begin{bmatrix} \begin{bmatrix} A & B & E \\ B & C & F \\ E & F & H \end{bmatrix} & \begin{bmatrix} B & C & F \\ C & D & G \\ F & G & J \end{bmatrix} & \begin{bmatrix} E & F & H \\ F & G & J \\ H & J & K \end{bmatrix} \end{bmatrix}$$

A First Look at Tensor Diagrams

The Solution

Three Kinds of Matrix

$$[\text{point}] \times \mathbf{T} = [\text{point}]$$

$$[\text{point}] \times \mathbf{Q} = [\text{line}]^T$$

$$[\text{line}]^T \times \mathbf{Q}^* = [\text{point}]$$

Old Index Types

$$\mathbf{P} = \begin{bmatrix} P_1 & P_2 & P_3 \end{bmatrix}$$

row

$$\mathbf{L} = \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix}$$

column

New Index Types

contravariant

$$\mathbf{P} = \begin{bmatrix} P^1 & P^2 & P^3 \end{bmatrix}$$

$$\mathbf{L} = \begin{bmatrix} L_1 & L_2 & L_3 \end{bmatrix}$$

covariant

Three Kinds of Matrix

$$T^i_j$$

Mixed

$$Q_{ij}$$

Pure covariant

$$\left(Q^*\right)^{ij}$$

Pure contravariant

The Multiplication Machine

$$\mathbf{P}\cdot\mathbf{L} = \begin{bmatrix} P_1 & P_2 & P_1 \end{bmatrix} \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix}$$

$$= P^1 L_1 + P^2 L_2 + P^3 L_3$$

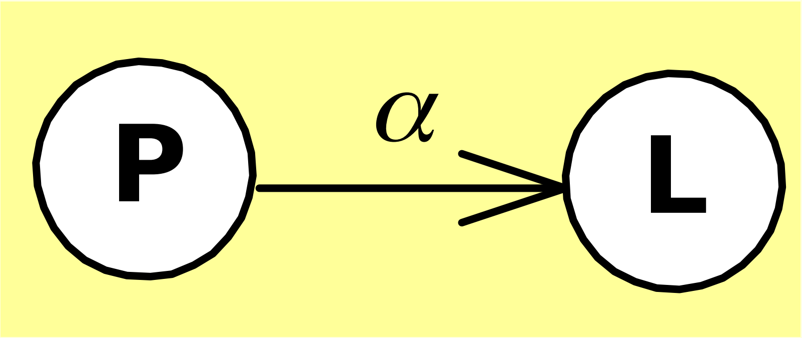
$$= \sum_i P^i L_i$$

$$= P^a E_a$$

Einstein Index Notation

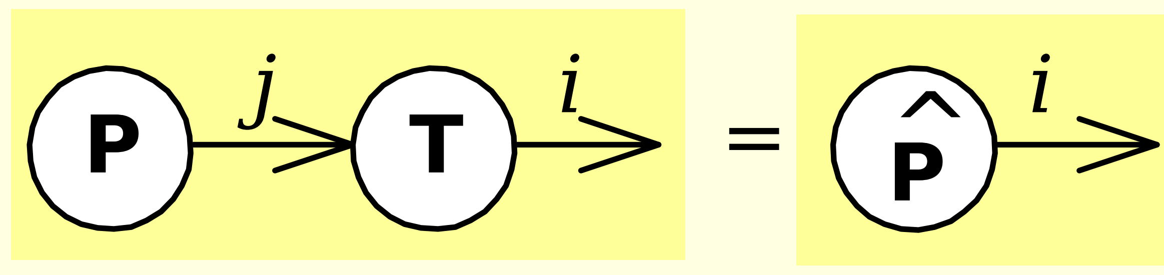
The Tensor Diagram of Dot Product

$$P^a L_a$$

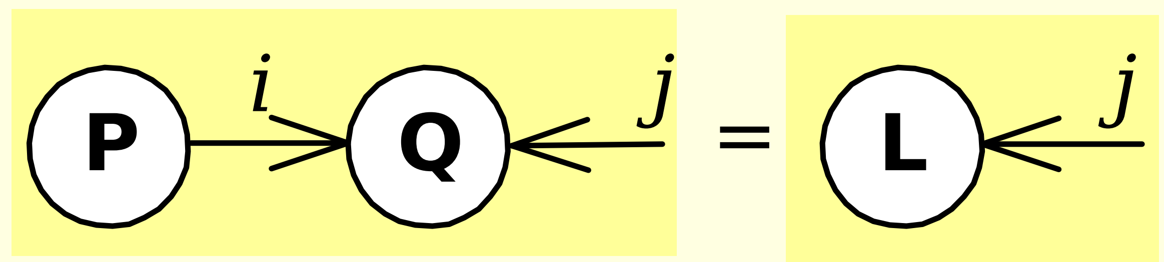


Three Kinds of Matrix

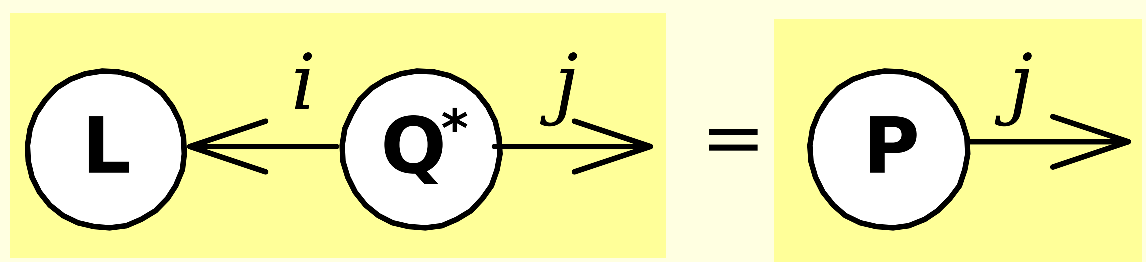
$$P^j T_j^i = \hat{P}^i$$



$$P^i Q_{ij} = L_j$$

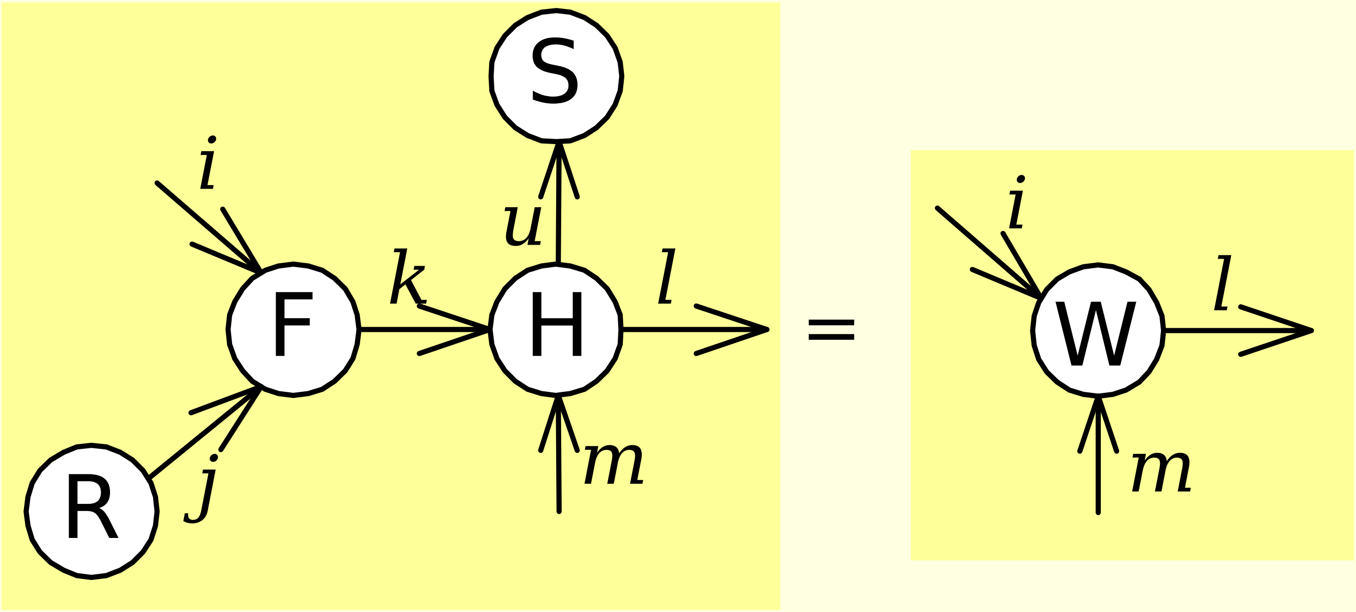


$$L_i \left(Q^*\right)^{ij} = P^j$$

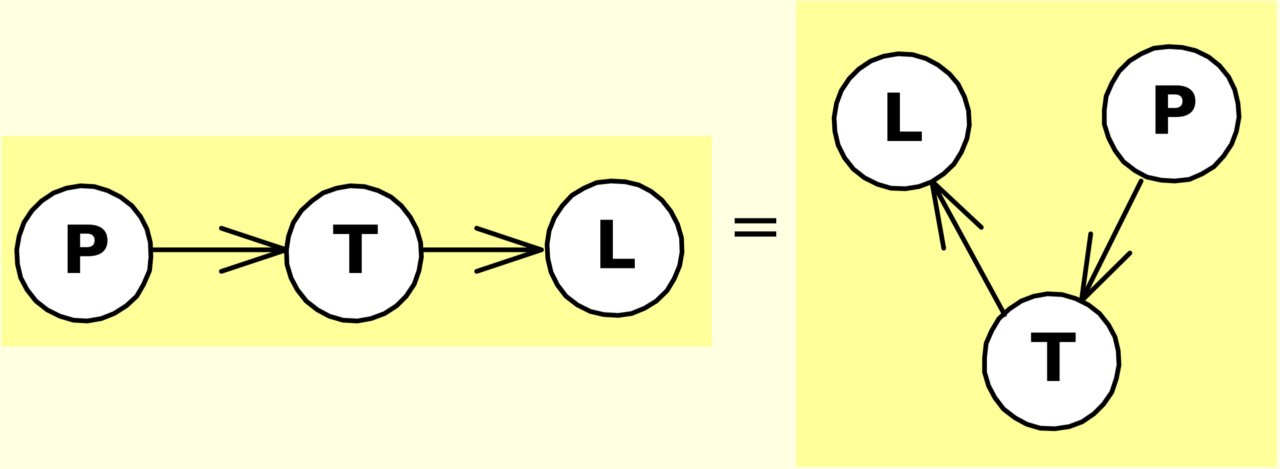


General Tensor Contraction

$$F_{ij}^{k} H_{km}^{lu} R^{j} S_{u} = W_{im}^{l}$$

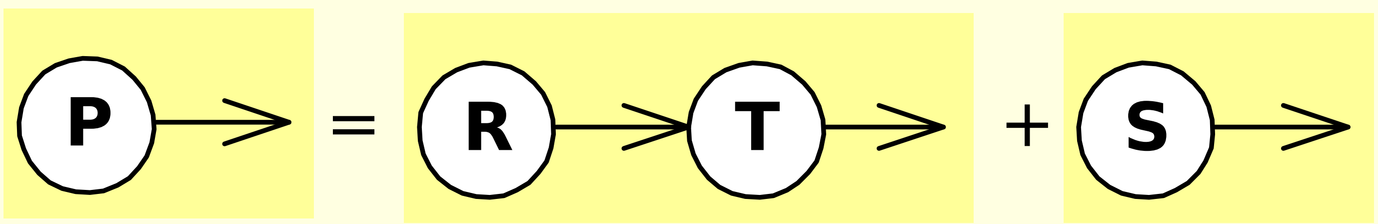


Rearranging Nodes Doesn't Change Value



Sum of Terms

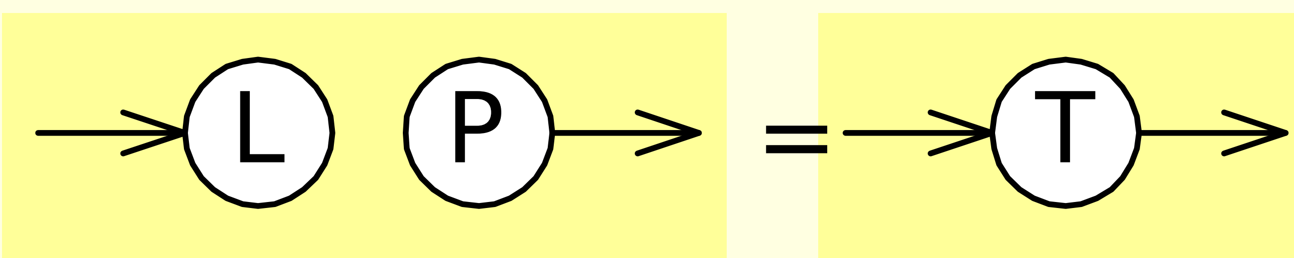
$$\mathbf{P} = \mathbf{R}\mathbf{T} + \mathbf{S}$$



Outer Product

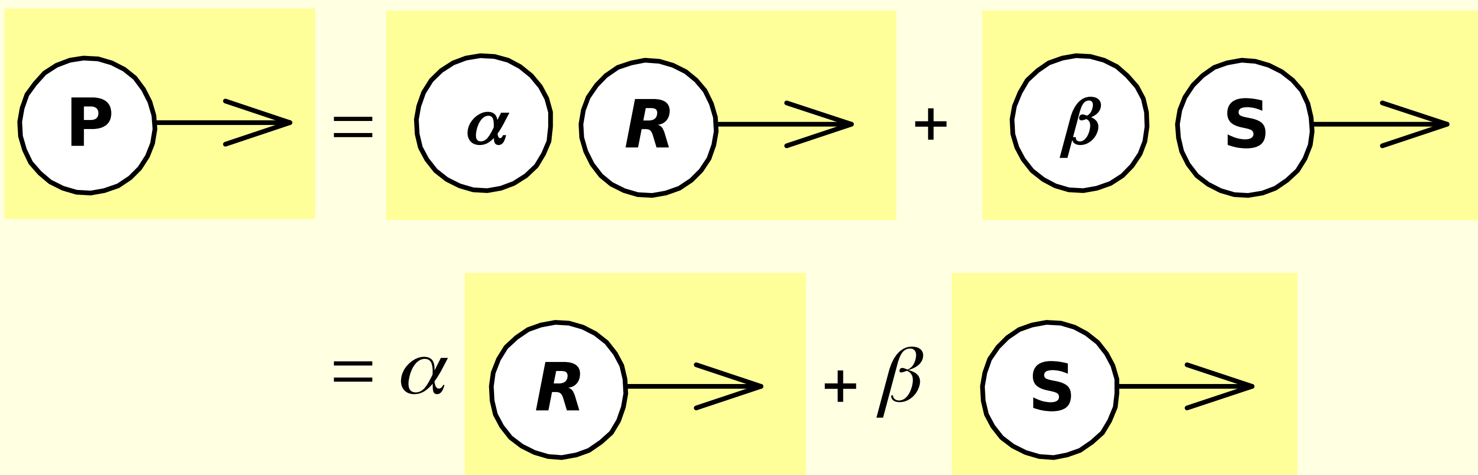
$$\begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} x & w \end{bmatrix} = \begin{bmatrix} ax & aw \\ bx & bw \end{bmatrix}$$

$$\mathbf{L}\,\mathbf{P} = \mathbf{T}$$



Scalar Product

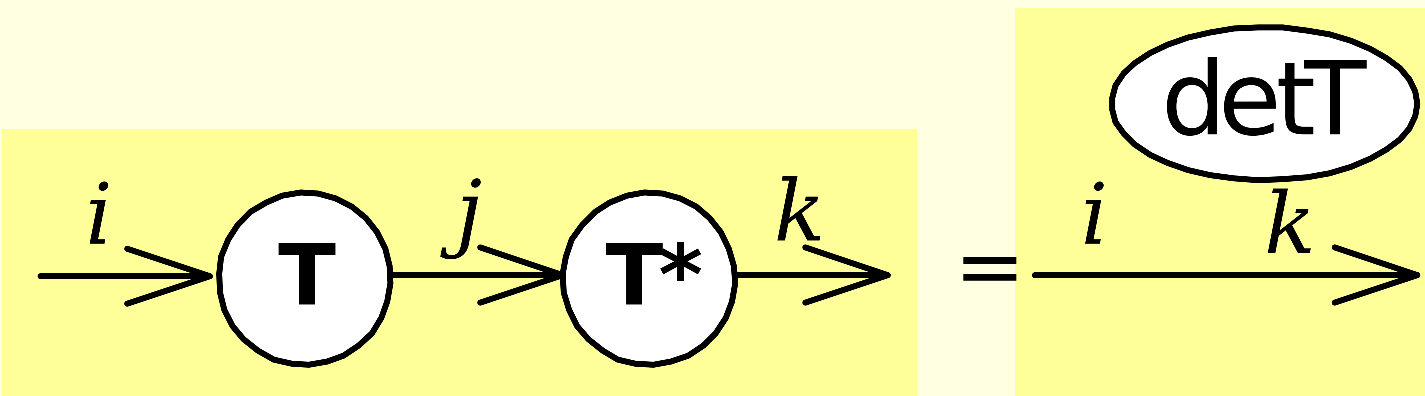
$$\mathbf{P} = a\mathbf{R} + b\mathbf{S}$$



Adjoint (of mixed tensor)

$$\mathbf{T}\mathbf{T}^* = \left(\det\mathbf{T}\right)\mathbf{I}$$

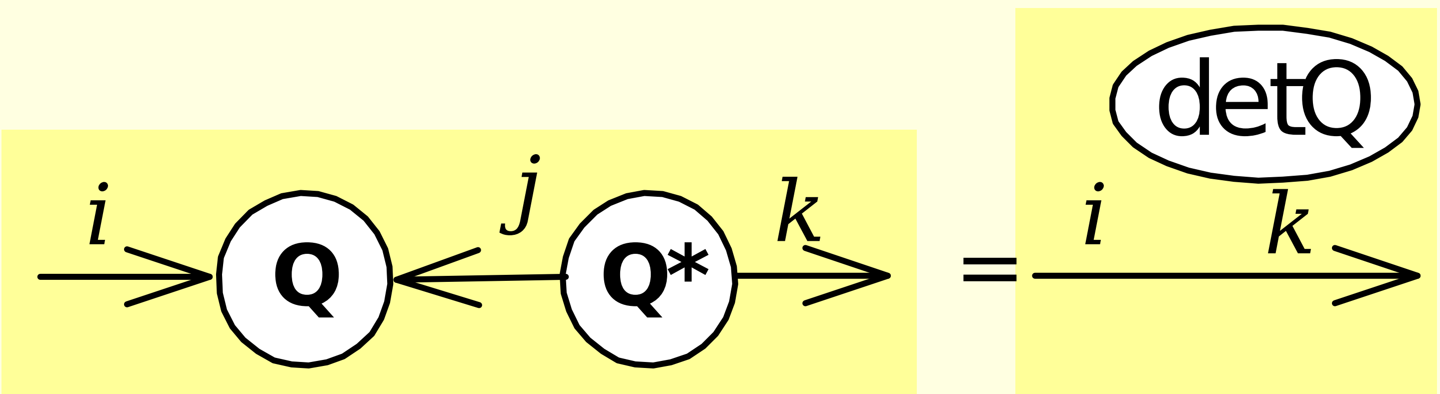
$$T_i^j \left(T^*\right)_j^k = \left(\det\mathbf{T}\right) d_i^k$$



Adjoint (of covariant tensor)

$$\mathbf{Q}\mathbf{Q}^{*} = \left(\det \mathbf{Q}\right)\mathbf{I}$$

$$Q_{i,j}\left(Q^{*}\right)^{j,k} = \left(\det \mathbf{T}\right) d_{i}^{k}$$



Point on a Line

$$ax + by + cw = 0$$

$$\begin{bmatrix} x & y & w \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$$\mathbf{P} \times \mathbf{L} = 0$$

$$P^i L_i = 0$$

$$\boxed{P \xrightarrow{i} L} = 0$$

Point on a Quadratic Curve

$$
\begin{aligned}
Ax^2 + 2Bxy + 2Cxw \\
+ Dy^2 + 2Eyw \\
+ Fw^2 = 0
\end{aligned}
\qquad
\begin{bmatrix} x & y & w \end{bmatrix}
\begin{bmatrix} A & B & C \\ B & D & E \\ C & E & F \end{bmatrix}
\begin{bmatrix} x \\ y \\ w \end{bmatrix} = 0
$$

$$\mathbf{P} \cdot \mathbf{Q} \cdot \mathbf{P}^T = 0$$

$$P^i Q_{ij} P^j = 0$$

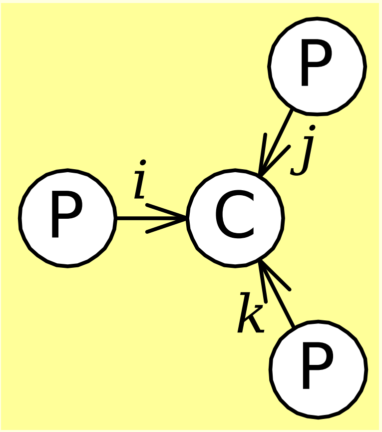
$$\text{(P)} \xrightarrow{i} \text{(Q)} \xleftarrow{j} \text{(P)} = 0$$

Point on a Cubic Curve

$$\begin{aligned}
&Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3 \\
&+3Ex^2w + 6Fxyw + 3Gy^2w \\
&+3Hxw^2 + 3Jyw^2 \\
&+Kw^3 = 0
\end{aligned}$$

$$\left\{ \begin{bmatrix} x & y & w \end{bmatrix} \begin{bmatrix} \begin{bmatrix} A & B & E \\ B & C & F \\ E & F & H \end{bmatrix} & \begin{bmatrix} B & C & F \\ C & D & G \\ F & G & J \end{bmatrix} & \begin{bmatrix} E & F & H \\ F & G & J \\ H & J & K \end{bmatrix} \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} \right\} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = 0$$

$$P^i P^j P^k C_{ijk} = 0$$

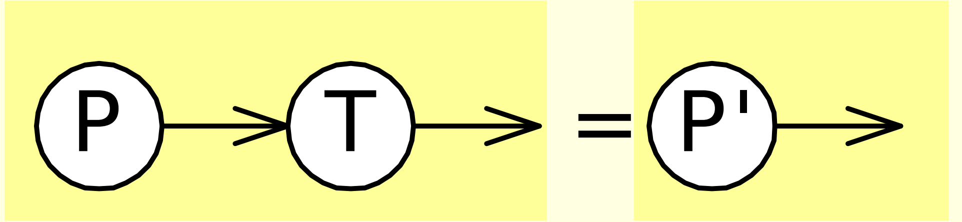


$= 0$

Transforming a Point

$$\mathbf{P}\mathbf{T} = \mathbf{P}'$$

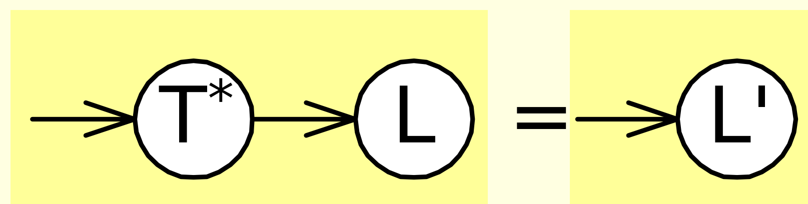
$$P^i T_i^j = \left(P'\right)^j$$



Transforming a Line

$$\left(\mathbf{T}^*\right)\mathbf{L} = \mathbf{L}'$$

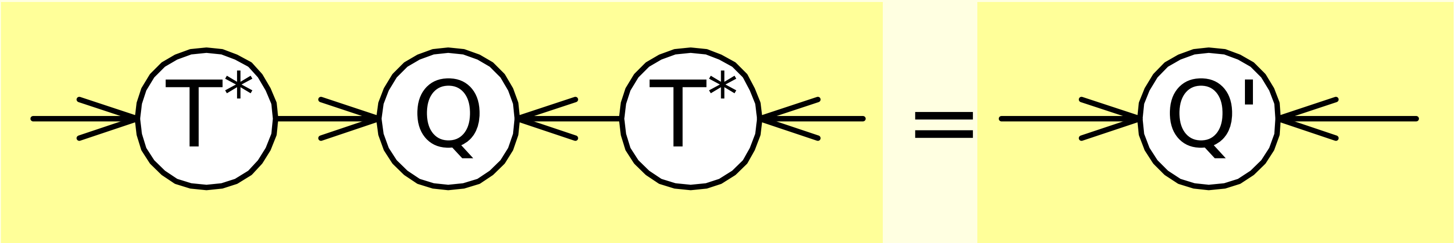
$$\left(T^*\right)^i_j L_i = \left(L'\right)_j$$



Transforming A Quadratic Curve

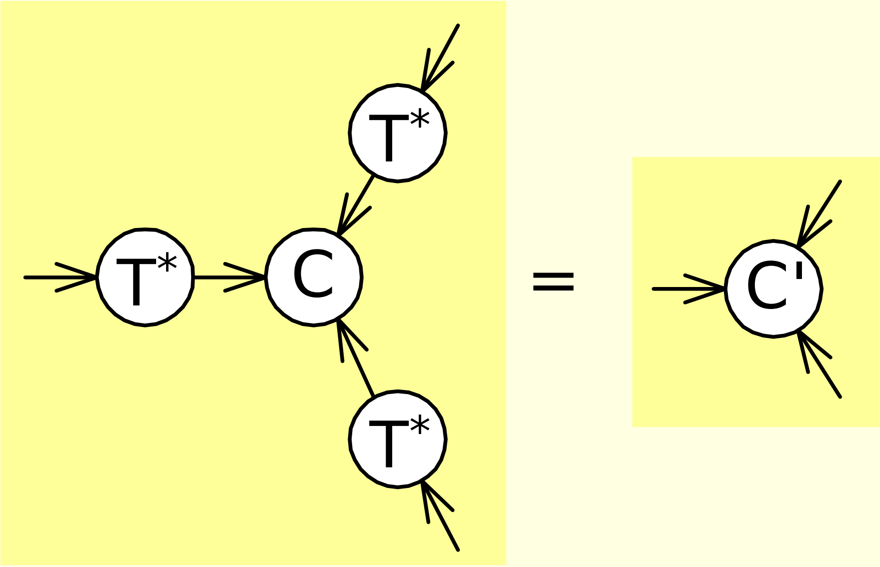
$$\left(\mathbf{T}^*\right)\mathbf{Q}\left(\mathbf{T}^*\right)^T = \mathbf{Q}'$$

$$\left(T^*\right)^i_k Q_{ij} \left(T^*\right)^j_l = \left(Q'\right)_{kl}$$



Transforming a Cubic Curve

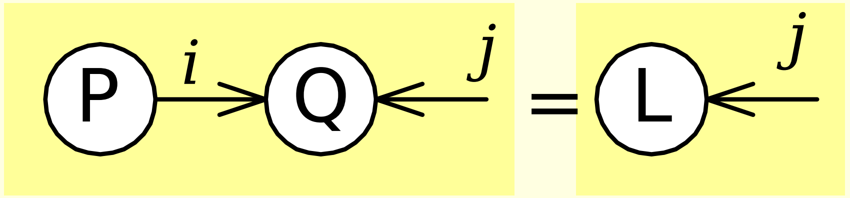
$$\left(T^{*}\right)_{l}^{i}\left(T^{*}\right)_{m}^{j}\left(T^{*}\right)_{n}^{k} C_{ijk} = \left(\hat{C}\right)_{lmn}$$



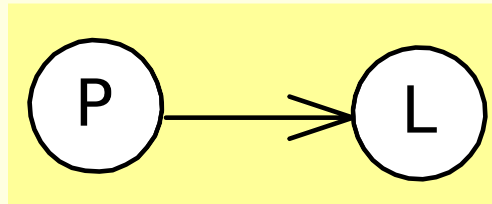
Tangent to Quadratic Curve

$$\mathbf{P}\times\mathbf{Q}=\mathbf{L}^{T}$$

$$P^{i}Q_{ij}=L_{j}$$



Dimensionality in Diagrams

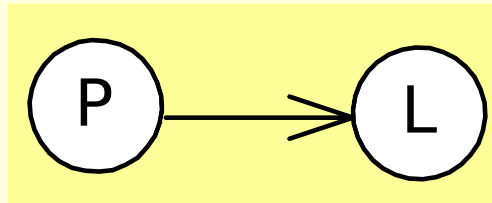


$$2D: \quad P^1 L_1 + P^2 L_2$$

$$3D: \quad P^1 L_1 + P^2 L_2 + P^3 L_3$$

$$4D: \quad P^1 L_1 + P^2 L_2 + P^3 L_3 + P^4 L_4$$

Dimensionality in Diagrams

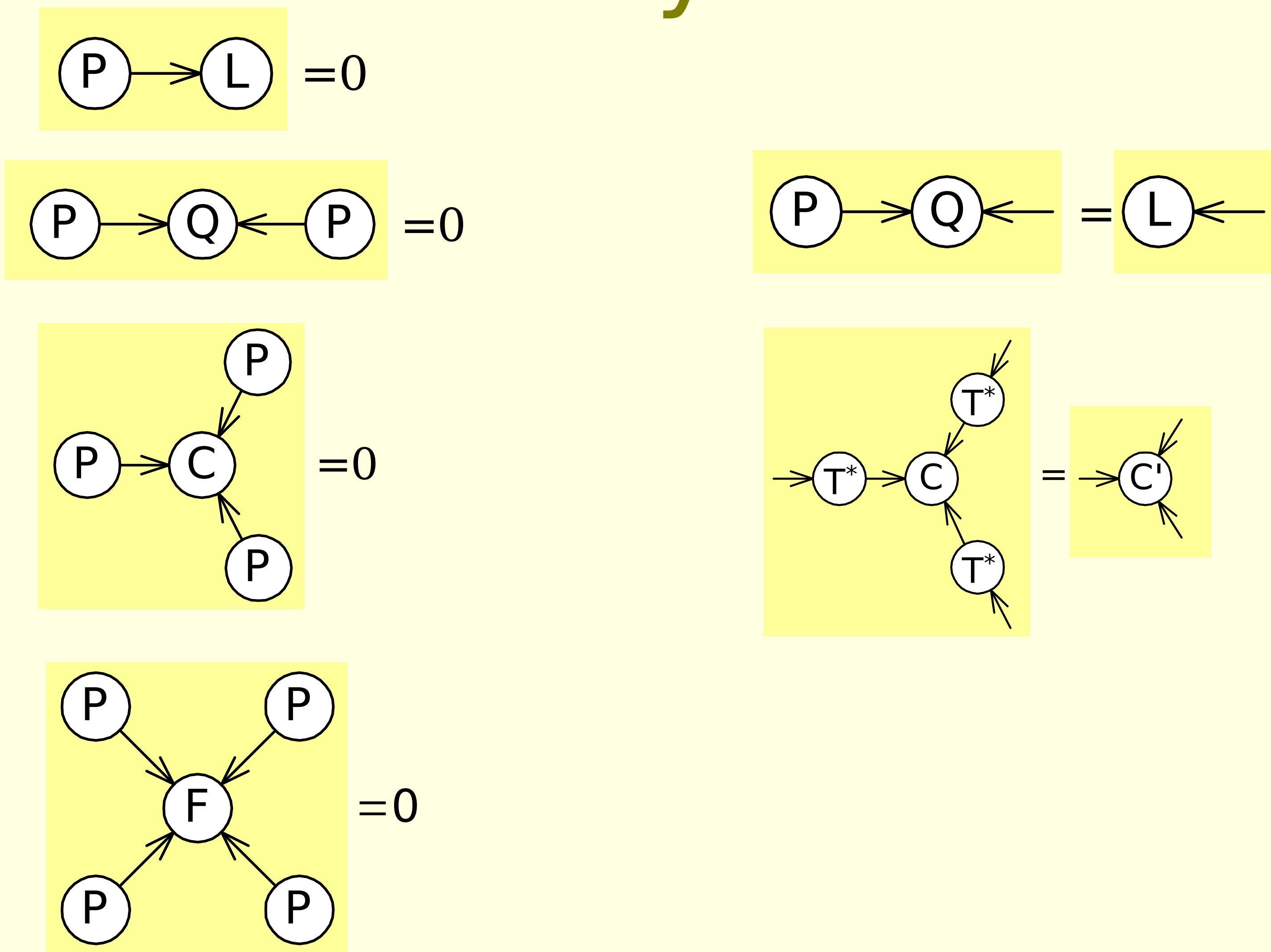


$$2D: \quad ax + bw$$

$$3D: \quad ax + by + cw$$

$$4D: \quad ax + by + cz + dw$$

Same Across Dimensionality



Changes with Dimensionality

- Cross Products
  - 3D (2DH)
  - 4D (3DH)
  - 2D (1DH)

3D (2DH)
Levi-Civita Epsilon

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = +1$$

$$\epsilon_{321} = \epsilon_{132} = \epsilon_{213} = -1$$

$$\epsilon_{ijk} = 0 \quad \text{otherwise}$$

$$\epsilon = \begin{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{bmatrix}$$

3D (2DH) Cross Product

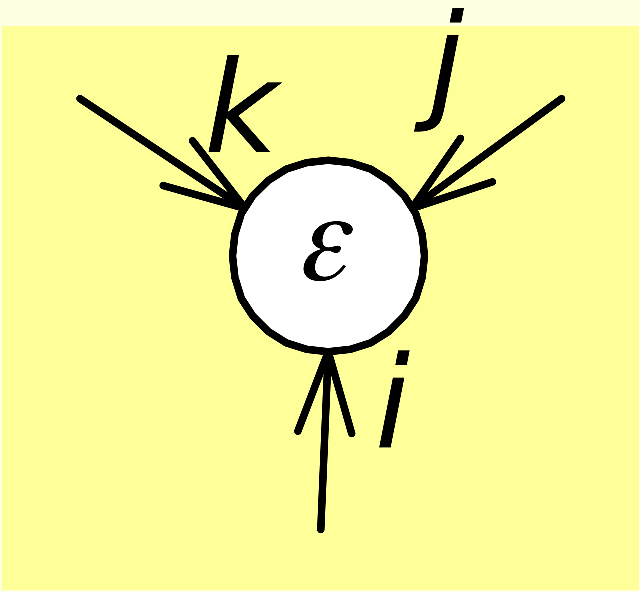
$$\begin{bmatrix} x_1 & y_1 & w_1 \end{bmatrix} \begin{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ w_2 \end{bmatrix} =$$

$$\begin{bmatrix} y_1 w_2 - w_1 y_2 & w_1 x_2 - x_1 w_2 & y_1 x_2 - x_1 y_2 \end{bmatrix}$$

$$\left(P1 \right)^i \left(P2 \right)^j e_{ijk} = L_k$$

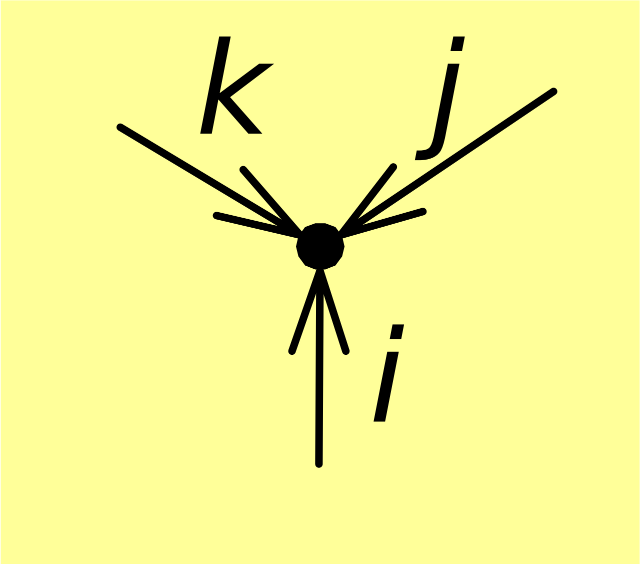
Levi-Civita Epsilon Diagram

$$e_{ijk}$$



Levi-Civita Epsilon Diagram

$$e_{ijk}$$

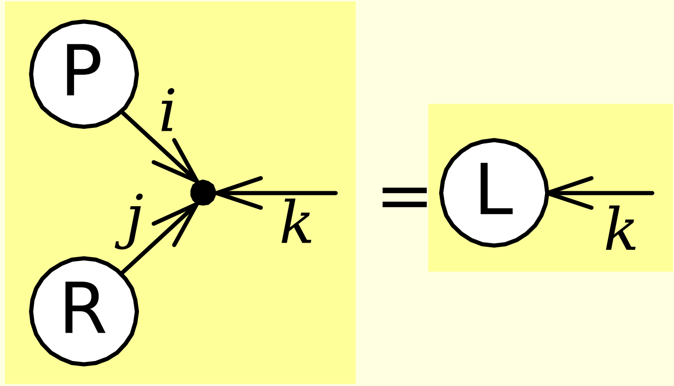


Cross Product

$$\begin{bmatrix}P^1 & P^2 & P^3\end{bmatrix} \times \begin{bmatrix}R^1 & R^2 & R^3\end{bmatrix} = \begin{bmatrix}L_1 \\ L_2 \\ L_3\end{bmatrix}$$

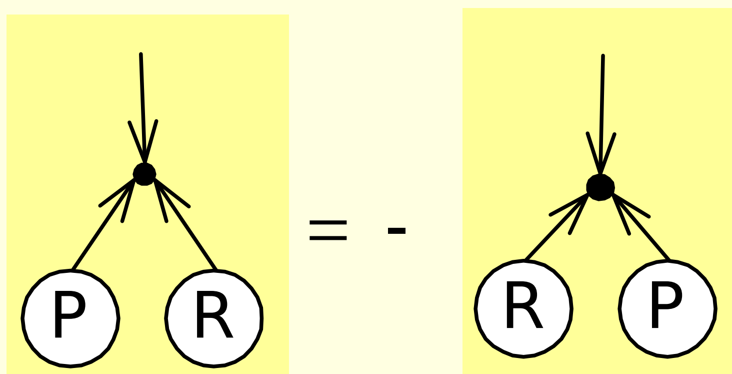
$$\mathbf{P} \times \mathbf{R} = \mathbf{L}$$

$$P^i R^j \epsilon_{ijk} = L_k$$



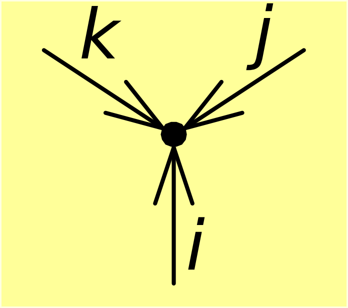
Anti-Symmetry and Epsilon

$$\mathbf{P} \times \mathbf{R} = -\,\mathbf{R} \times \mathbf{P}$$

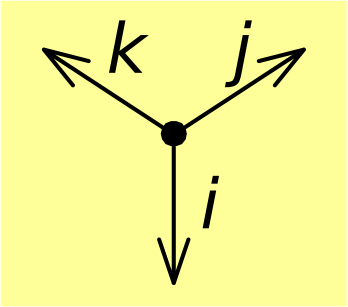


Levi-Civita Epsilon

COvariant $\quad e_{ijk}$



CONTRAvariant $\quad e^{ijk}$

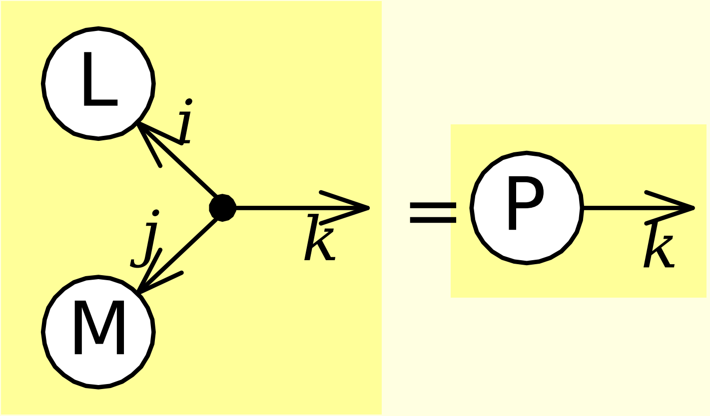


The Other Cross Product

$$\begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} \times \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix} = \begin{bmatrix} P^1 & P^2 & P^3 \end{bmatrix}$$

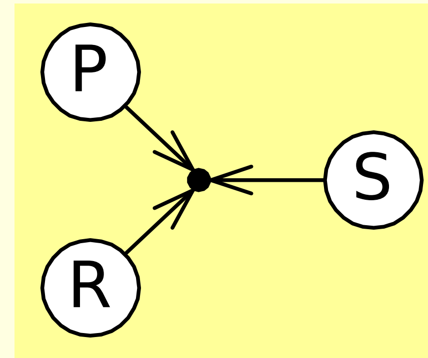
$$\mathbf{L} \times \mathbf{M} = \mathbf{P}$$

$$L_i M_j e^{ijk} = P^k$$

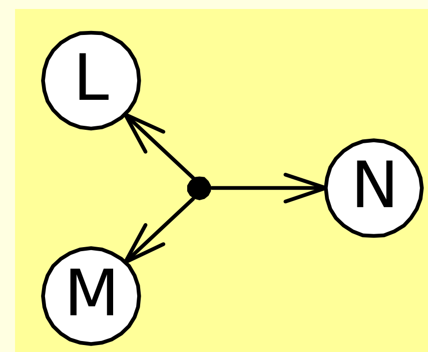


Triple Product

$$\mathbf{P}\text{ ´ }\mathbf{R}\times\mathbf{S}=\mathbf{R}\text{ ´ }\mathbf{S}\times\mathbf{P}=\mathbf{S}\text{ ´ }\mathbf{P}\times\mathbf{R}$$



$$\mathbf{L}\text{ ´ }\mathbf{M}\times\mathbf{N}=\mathbf{M}\text{ ´ }\mathbf{N}\times\mathbf{L}=\mathbf{N}\text{ ´ }\mathbf{L}\times\mathbf{M}$$

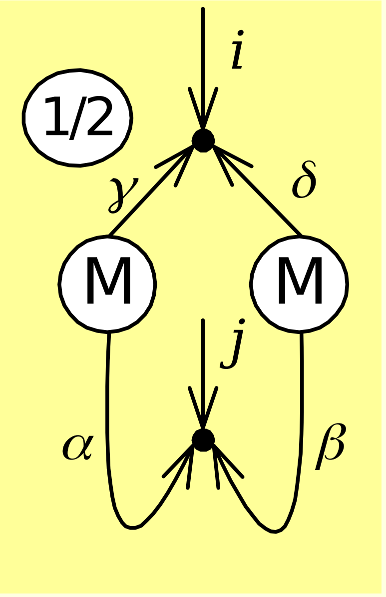


Adjoint of Matrix

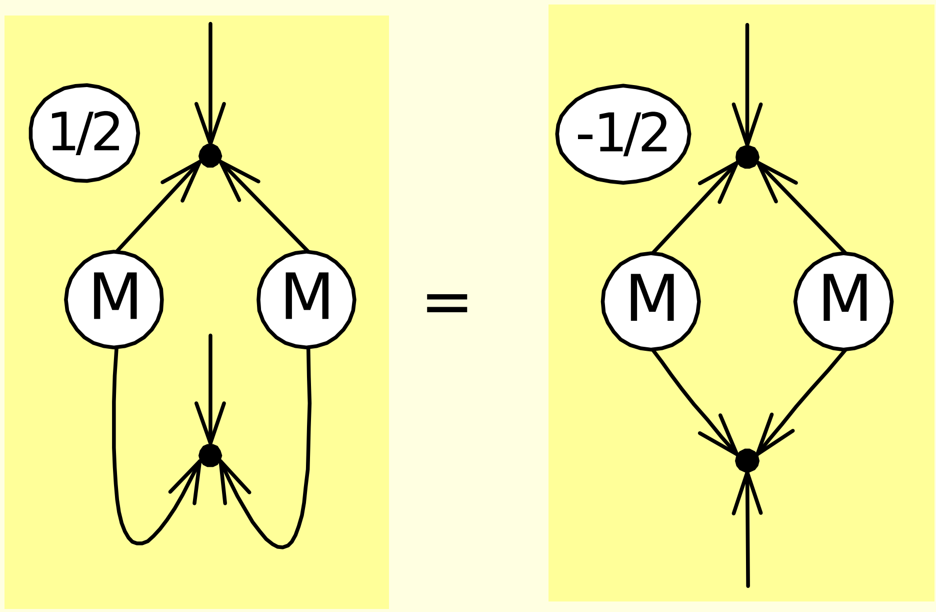
$$\begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \Rightarrow \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & M^{*23} & \cdot \end{bmatrix}$$

$$\left(M^{*}\right)^{23} = M_{21}M_{13} - M_{11}M_{23}$$

$$\left(M^{*}\right)^{ji} = \tfrac{1}{2}\varepsilon^{j\alpha\beta}\varepsilon^{i\gamma\delta}M_{\alpha\gamma}M_{\beta\delta}$$

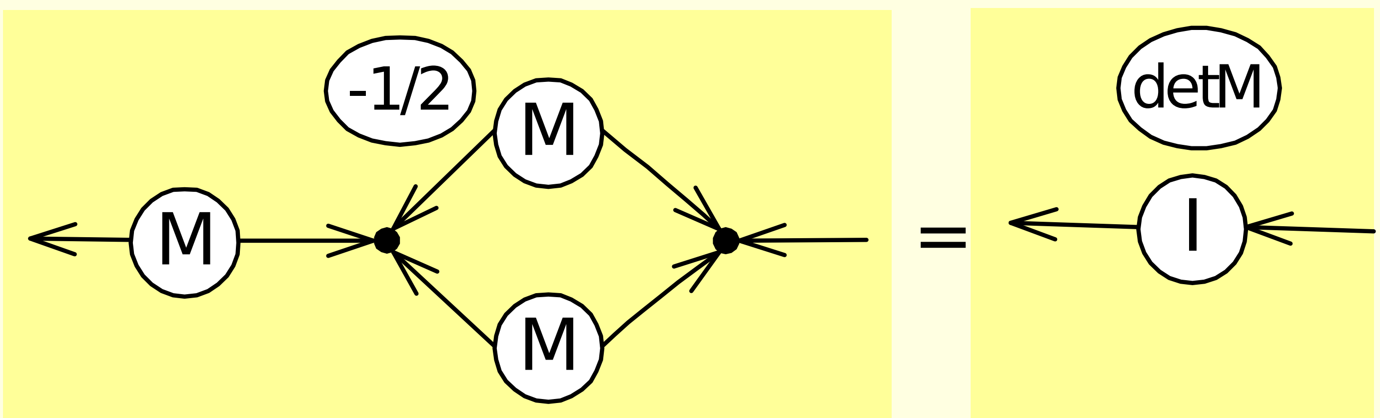


Adjoint of Matrix



Determinant of Matrix

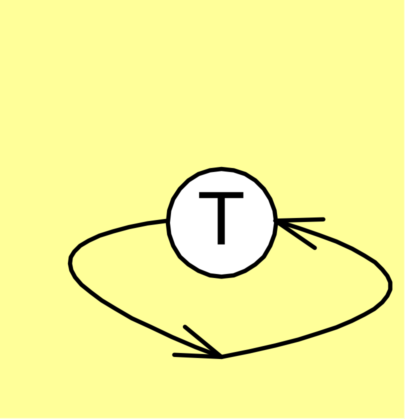
$$\mathbf{M}\mathbf{M}^{*} = \left(\det \mathbf{M}\right)\mathbf{I}$$



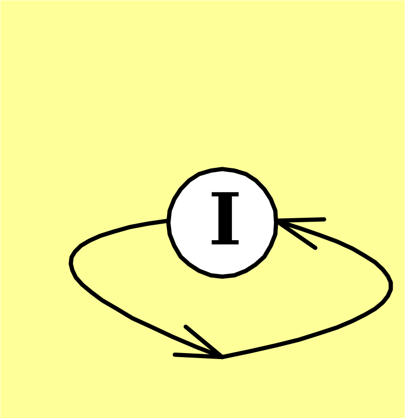
Trace of Matrix

$$\mathrm{trace}\,\mathbf{T} = \sum_i T^i_i$$

trace T =

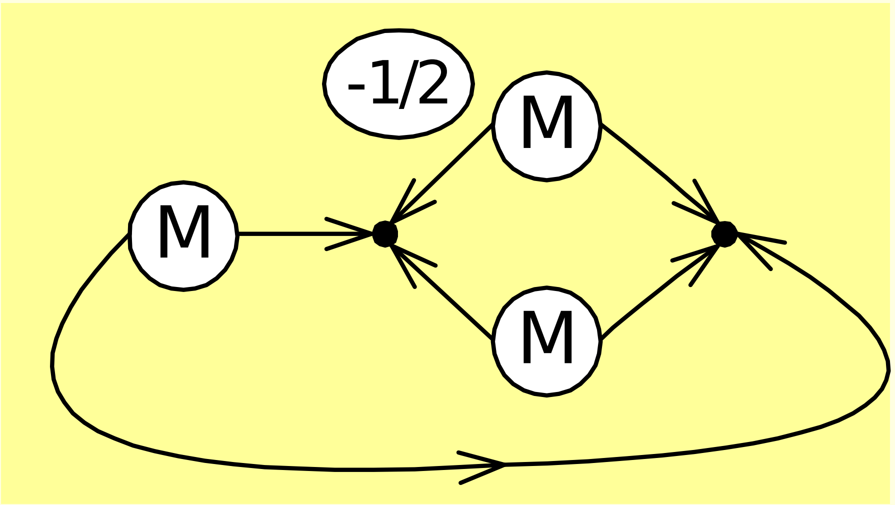


trace **I** =

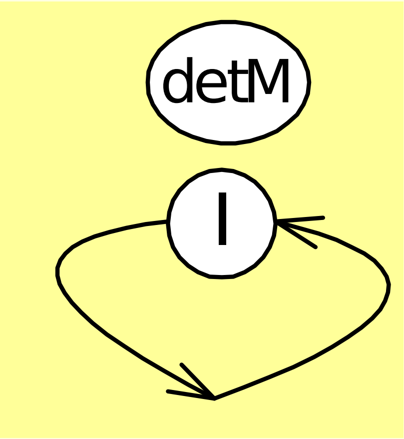


$= 3$

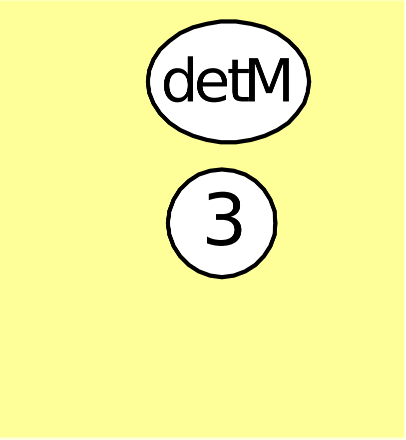
Determinant of Matrix



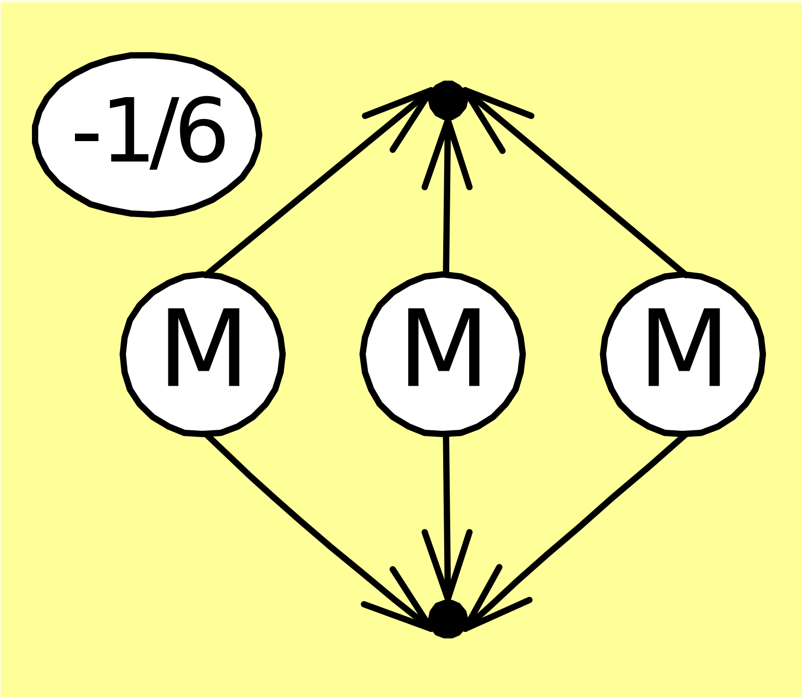
=



=



Determinant of Matrix



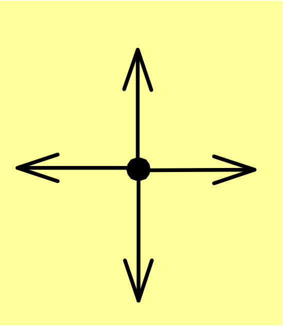
$$= \det \mathbf{M}$$

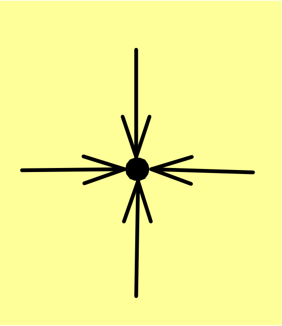
Levi-Civita Epsilon 4D (3DH)

$$e_{ijkl} = +1 \quad \text{if } ijkl \text{ is an even permutation of } 1234$$

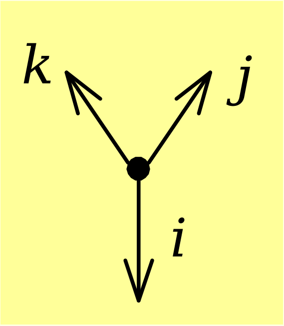
$$e_{ijkl} = -1 \quad \text{if } ijkl \text{ is an odd permutation of } 1234$$

$$e_{ijkl} = 0 \quad \text{otherwise}$$

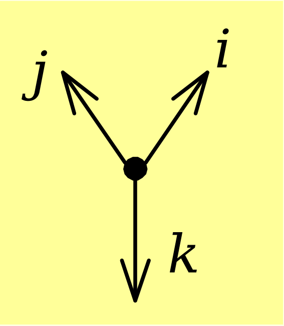




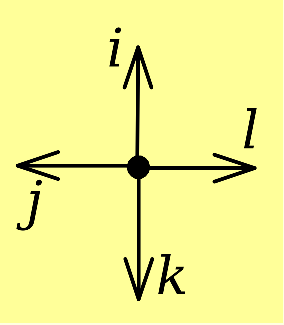
Anti-Symmetry of Epsilons



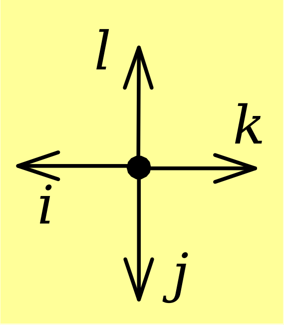
ε_{ijk}



ε_{kij}

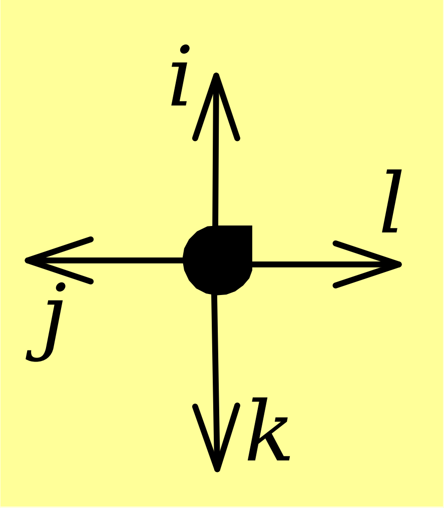


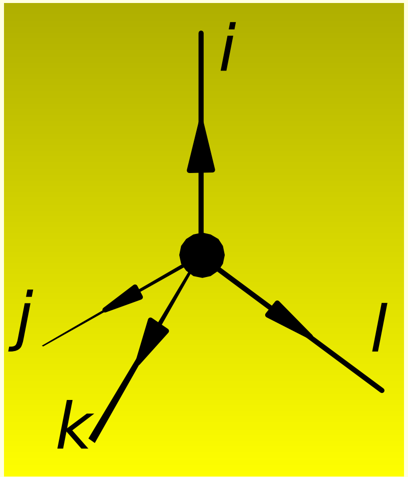
ε_{ijkl}



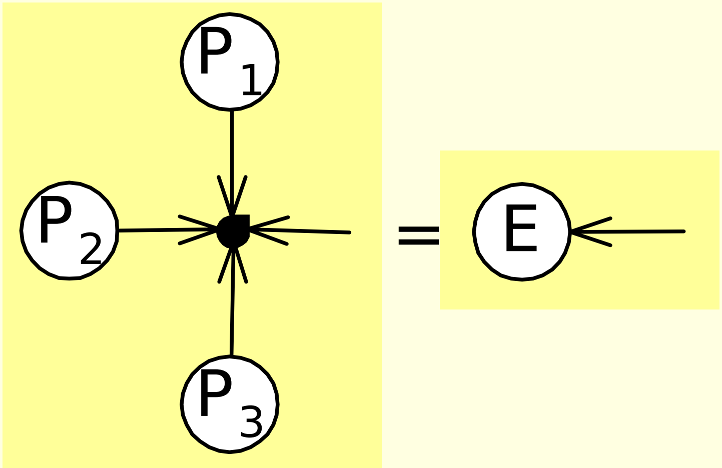
$-\ \varepsilon_{lijk}$

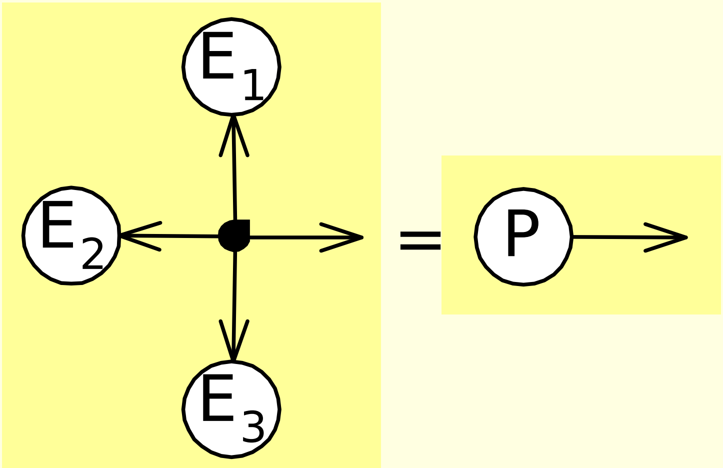
Notation for Anti-Symmetry of 4D Epsilon



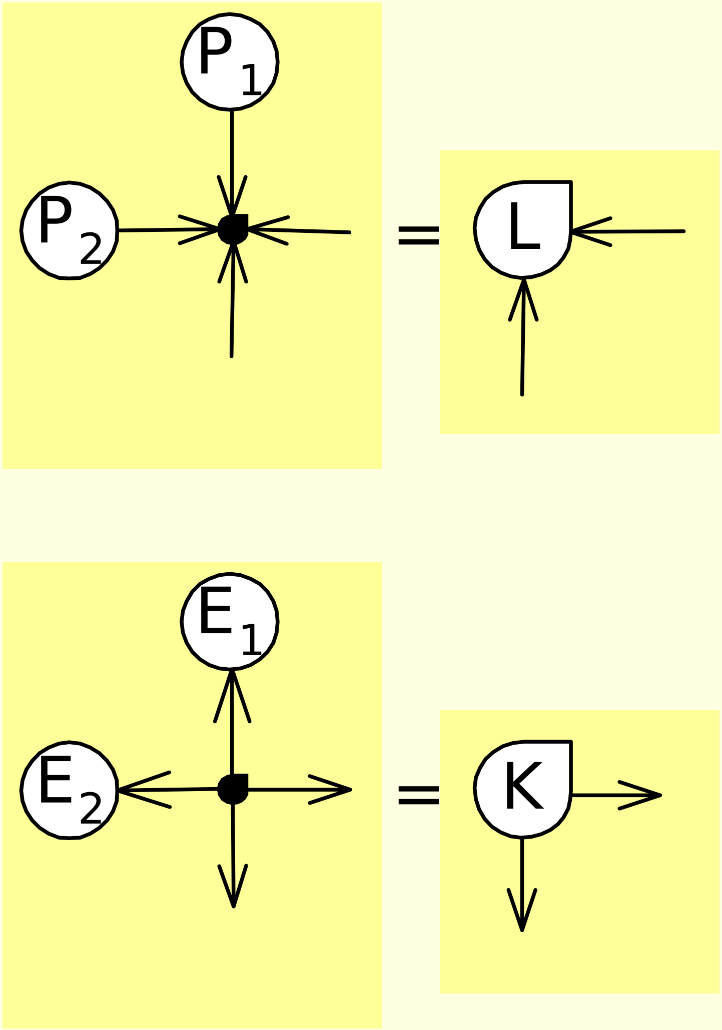


3 Points and 3 Planes

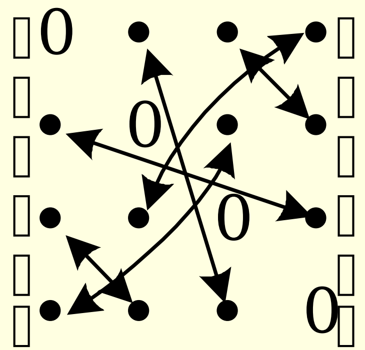


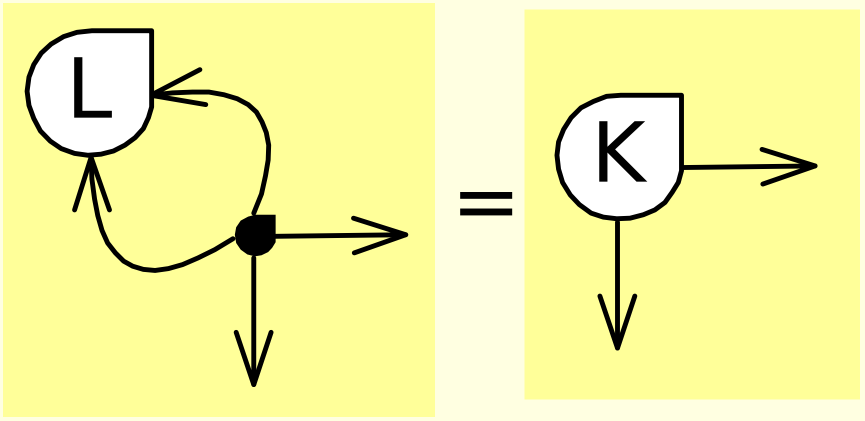


2 Points and 2 Planes = Line

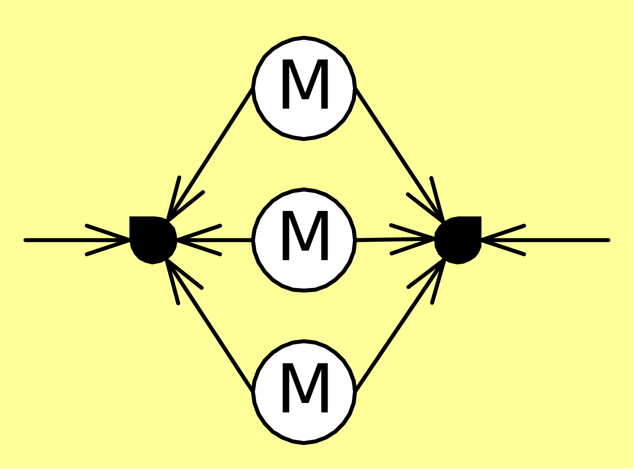


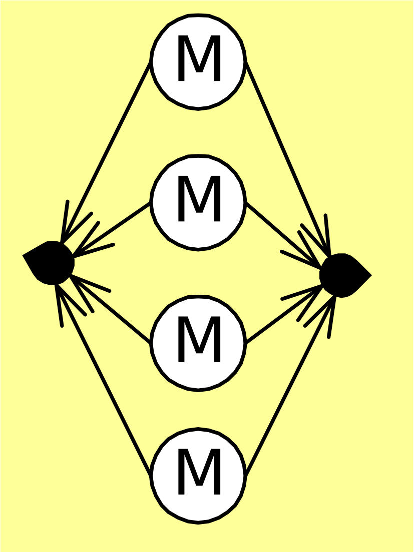
Relation between 2 Line Tensors





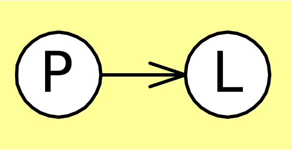
Adjoint and Determinant



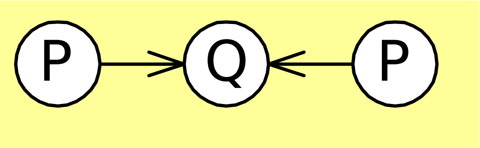


2D (1DH) Homogeneous Polynomials

$$Ax + Bw = \begin{bmatrix} x & w \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}$$

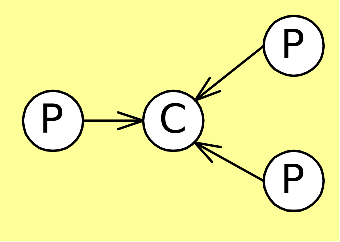


$$Ax^2 + 2Bxw + Cw^2 = \begin{bmatrix} x & w \end{bmatrix} \begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}$$

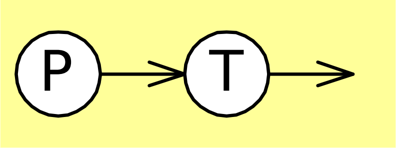


$$Ax^3 + 3Bx^2w + 3Cxw^2 + Dw^3$$

$$= \begin{bmatrix} x & w \end{bmatrix} \begin{bmatrix} \begin{bmatrix} A & B \\ B & C \end{bmatrix} & \begin{bmatrix} B & C \\ C & D \end{bmatrix} \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}$$



$$\begin{bmatrix} x & w \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} x' & w' \end{bmatrix}$$



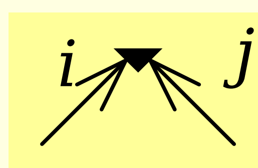
2D (1DH) Levi-Civita Epsilon

$$\epsilon_{12} = 1$$

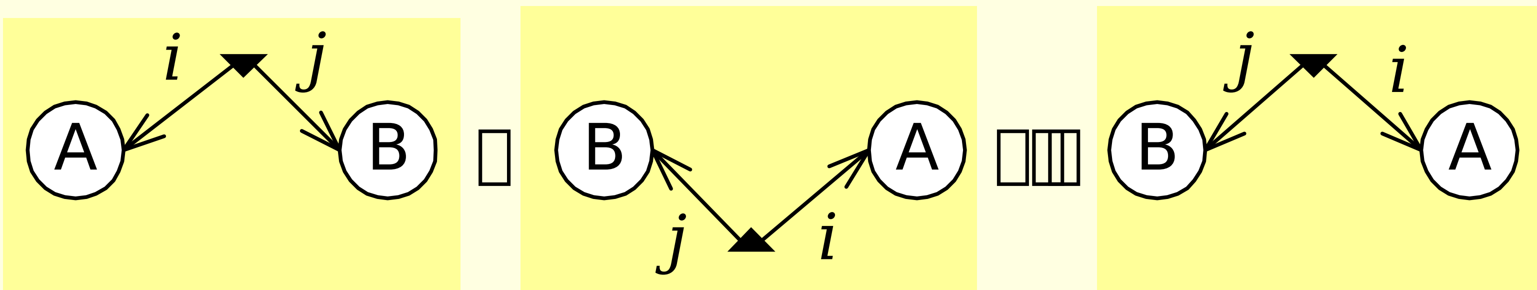
$$\epsilon_{21} = -1$$

$$\epsilon_{ij} = 0 \quad otherwise$$

$$\epsilon = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$



Anti-Symmetry of 2D Epsilon

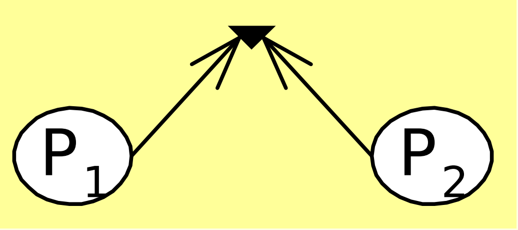


Homogeneous Equality

$$\frac{x_1}{w_1} = \frac{x_2}{w_2}$$

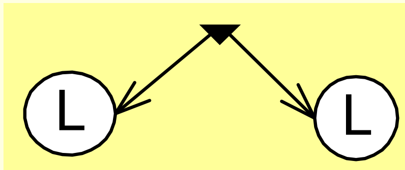
$$x_1 w_2 - w_1 x_2 = 0$$

$$\begin{bmatrix} x_1 & w_1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ w_2 \end{bmatrix} = 0$$

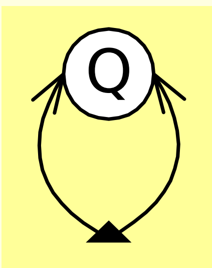


$= 0$

Identities

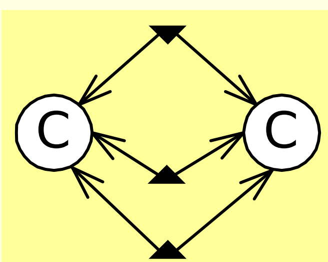


$= 0$

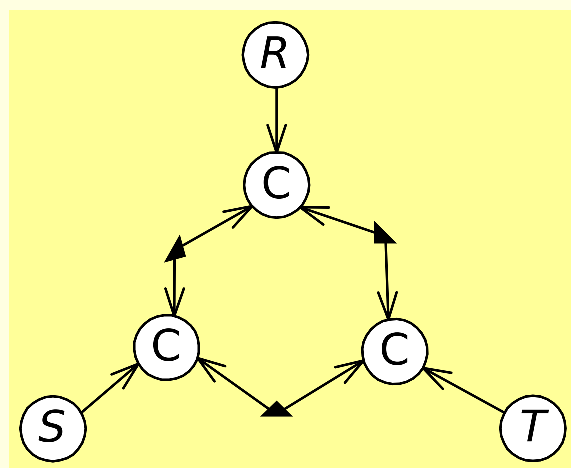


$= 0$

$$\mathrm{trace}\left(\begin{bmatrix} A & B \\ B & C \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right) = \mathrm{trace}\left(\begin{bmatrix} -B & A \\ -C & B \end{bmatrix}\right)$$

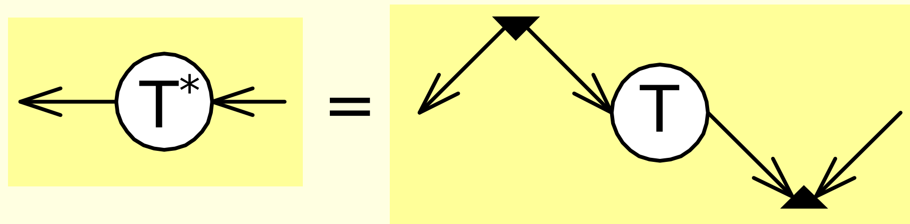


$= 0$

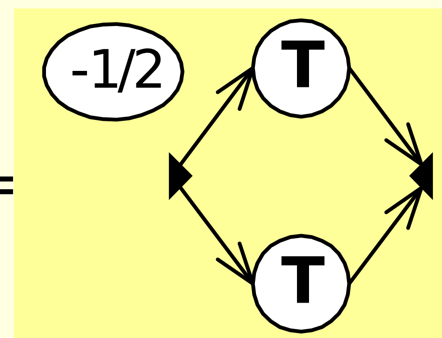


$= 0$

Adjoint and Determinant

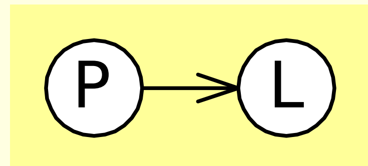


det T =



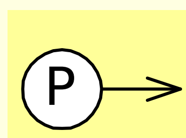
Solving Linear Equation

$$Ax + Bw = \begin{bmatrix} x & w \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} =$$

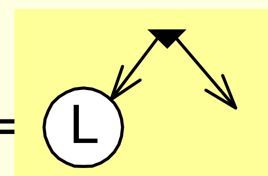


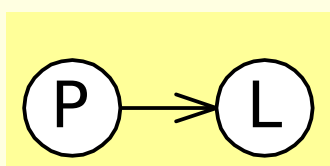
$$= 0$$

$$\begin{bmatrix} x & w \end{bmatrix} = \begin{bmatrix} -B & A \end{bmatrix}$$

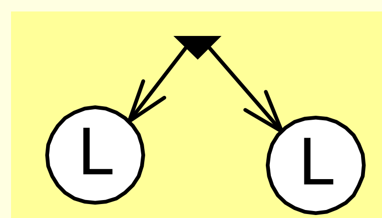


$$=$$



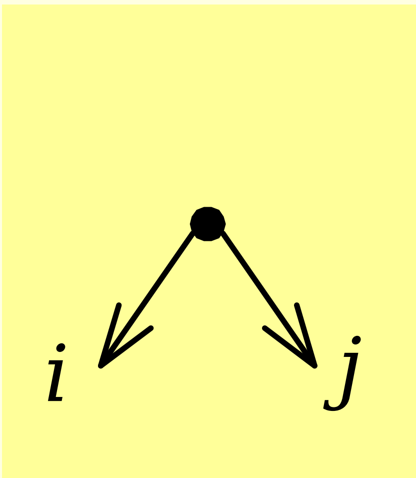


$$=$$



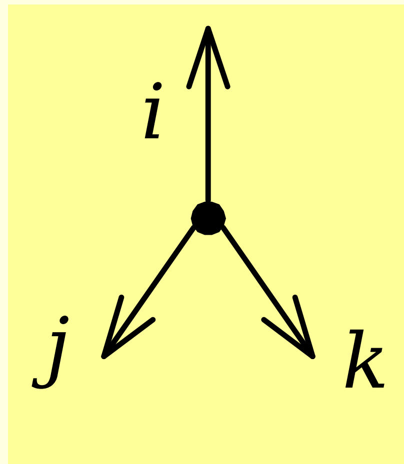
$$= 0$$

Dimensionality and Epsilon



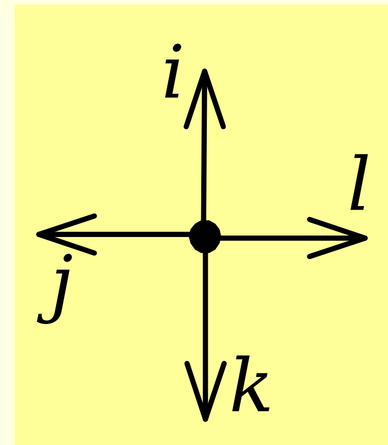
ε^{ij}

2D(1DH)



ε^{ijk}

3D(2DH)

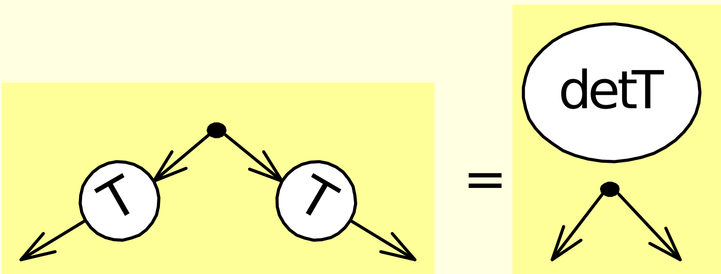


ε^{ijkl}

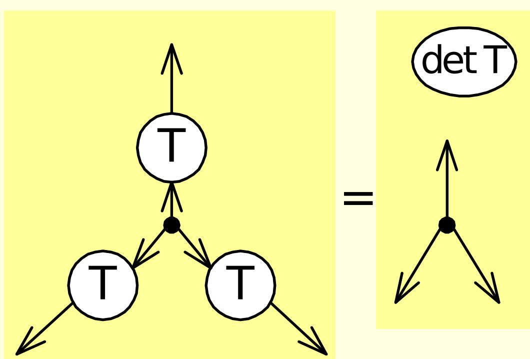
4D(3DH)

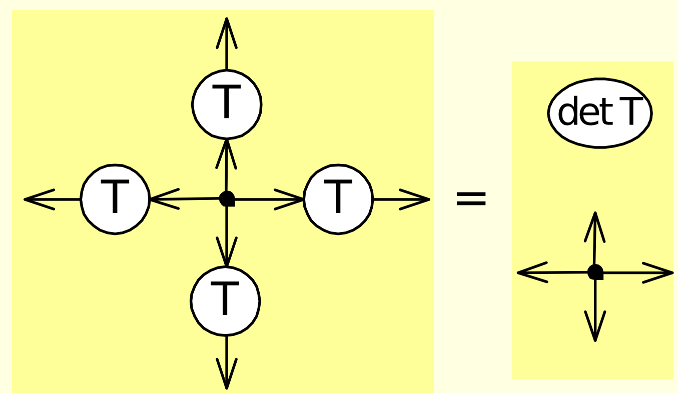
MAJOR PUNCHLINE

Another Determinant Identity

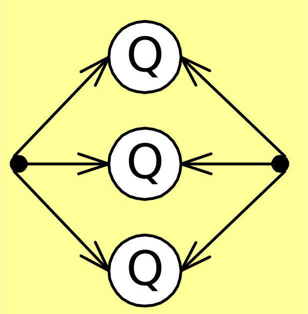


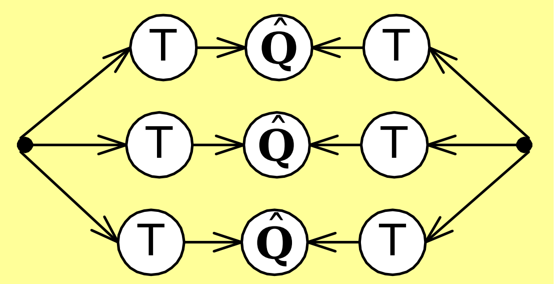
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 0 & ad-bc \\ bc-ad & 0 \end{bmatrix}$$

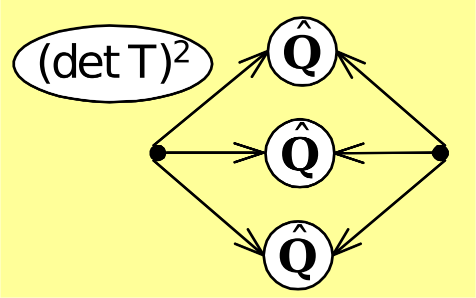




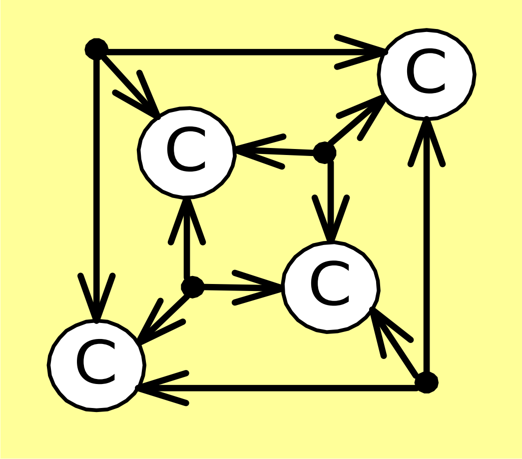
Transformationally Invariant Diagrams

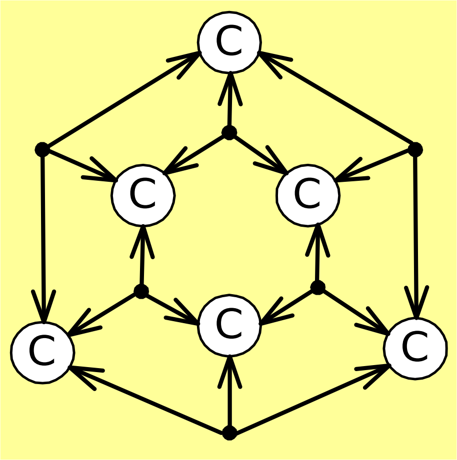






Invariants of Cubic Curve





The Epsilon-Delta Identity

The Basic Tool for Manipulating Tensor Diagrams

3D(2DH)
Epsilon-Delta Rule

$$e_{a,j,k}e^{a,l,m} = D^{l,m}_{j,k}$$

3D(2DH)
Epsilon-Delta Rule

$$e_{a,j,k}e^{a,l,m} = d^l_j d^m_k - d^m_j d^l_k$$

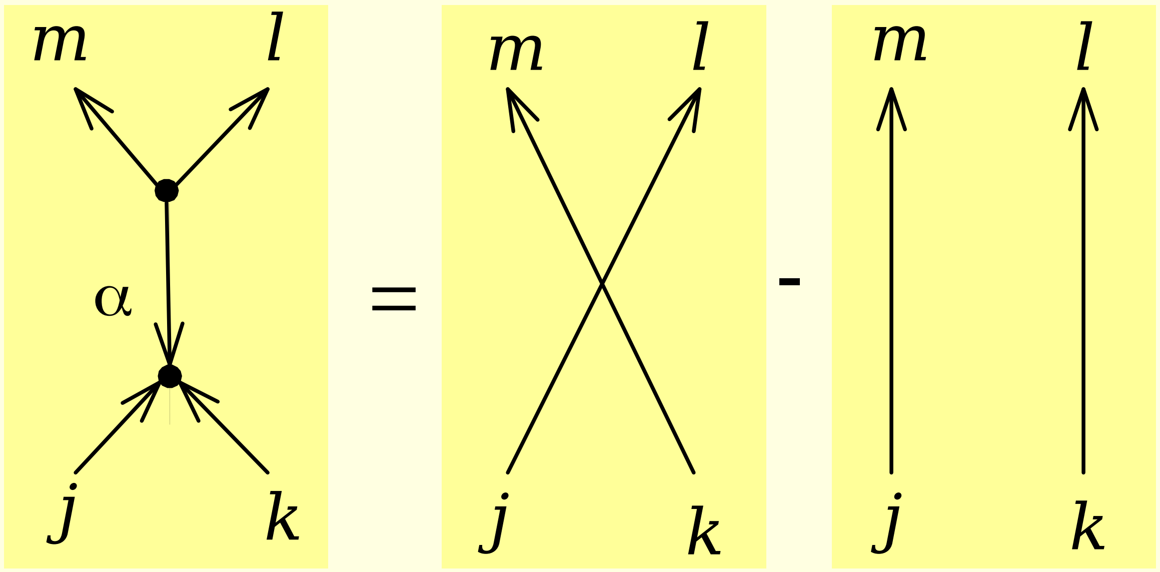
Kronecker Delta

$$d^i_j$$

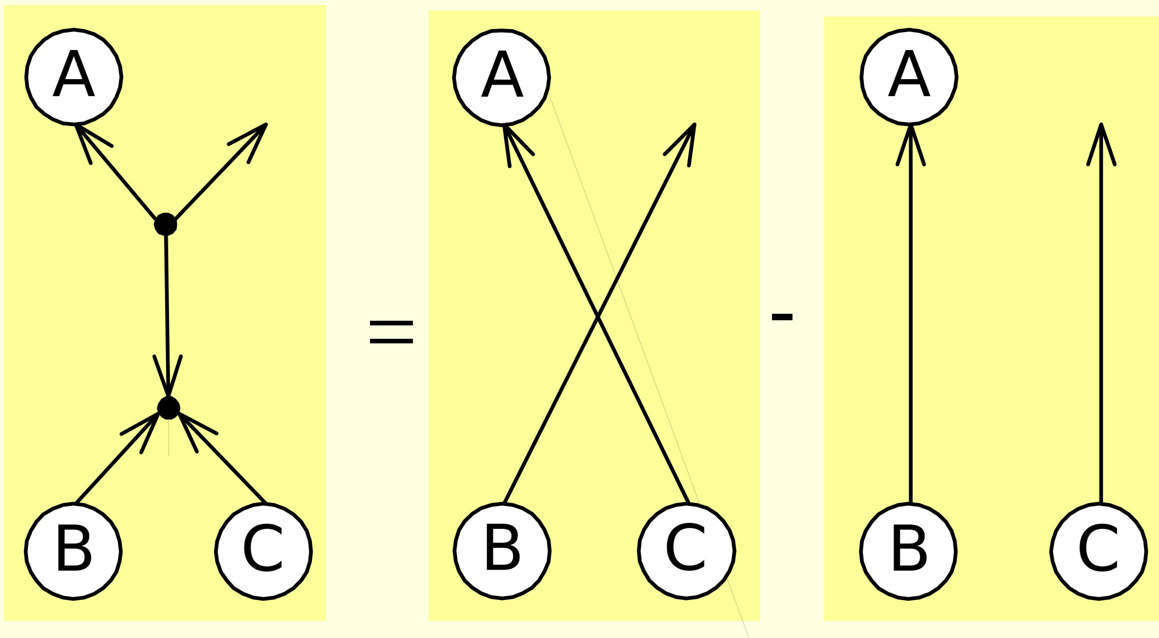
$$\xrightarrow{\quad j \qquad\qquad i \quad}$$

Epsilon-Delta Rule

$$\varepsilon_{\alpha jk}\varepsilon^{\alpha lm} = \delta^l_j \delta^m_k - \delta^m_j \delta^l_k$$

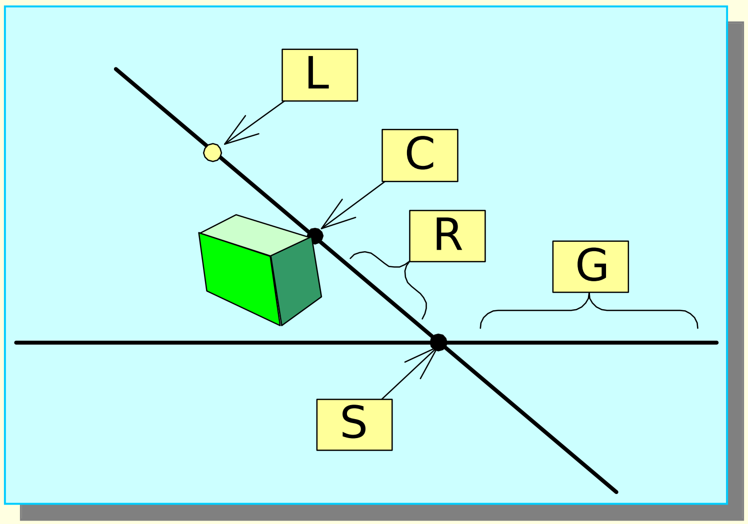


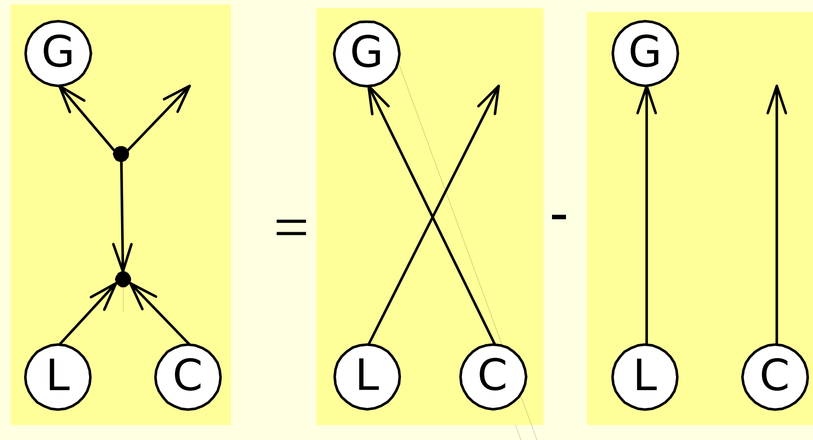
Epsilon-Delta Rule



$$A\times(B\times C)=(A\cdot C)B-(A\cdot B)C$$

Projection from L thru C onto G





$$\mathbf{L}\times\mathbf{C}=\mathbf{R}$$

$$\mathbf{G}\times\mathbf{R}=\mathbf{S}$$

$$\mathbf{S}=a\,\mathbf{L}+b\,\mathbf{C}$$

$$\mathbf{S}\cdot\mathbf{G}=0$$

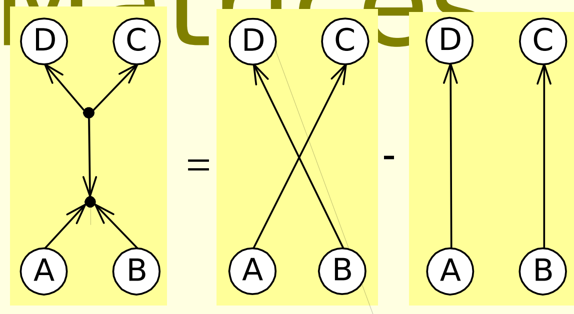
$$\left(a\,\mathbf{L}+b\,\mathbf{C}\right)\cdot\mathbf{G}=0$$

$$a\left(\mathbf{L}\cdot\mathbf{G}\right)+b\left(\mathbf{C}\cdot\mathbf{G}\right)=0$$

$$a=\left(\mathbf{C}\cdot\mathbf{G}\right),\ b=-\left(\mathbf{L}\cdot\mathbf{G}\right)$$

$$\mathbf{S}=\left(\mathbf{C}\cdot\mathbf{G}\right)\mathbf{L}-\left(\mathbf{L}\cdot\mathbf{G}\right)\mathbf{C}$$

Determinant of Product of Matrices



$$\begin{bmatrix}\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot\end{bmatrix}\begin{bmatrix}\cdot&\cdot\\ \cdot&\cdot\\ \cdot&\cdot\end{bmatrix}=\begin{bmatrix}\cdot&\cdot\\ \cdot&\cdot\end{bmatrix}$$

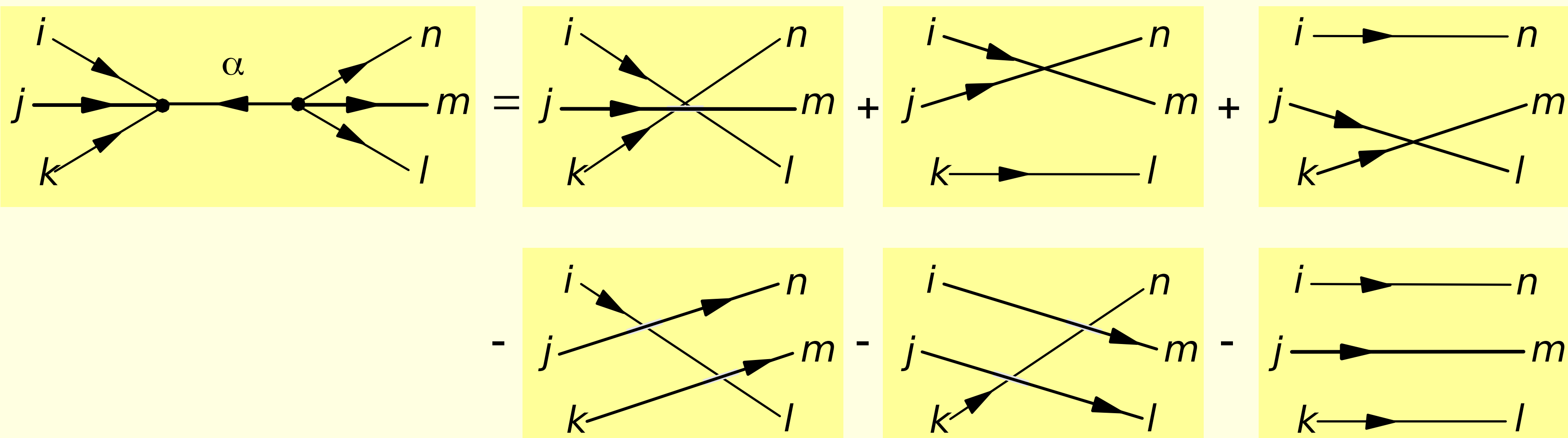
$$\begin{bmatrix}\cdots & \mathbf{A} & \cdots\\ \cdots & \mathbf{B} & \cdots\end{bmatrix}\begin{bmatrix}\vdots&\vdots\\ \mathbf{C} & \mathbf{D}\\ \vdots&\vdots\end{bmatrix}=\begin{bmatrix}\mathbf{A}\cdot\mathbf{C} & \mathbf{A}\cdot\mathbf{D}\\ \mathbf{B}\cdot\mathbf{C} & \mathbf{B}\cdot\mathbf{D}\end{bmatrix}=\mathbf{M}$$

$$\begin{bmatrix}\cdots & \mathbf{C}\times\mathbf{D} & \cdots\end{bmatrix}\begin{bmatrix}\vdots\\ \mathbf{A}\times\mathbf{B}\\ \vdots\end{bmatrix}=\det\mathbf{M}$$

4D (3DH) Epsilon Delta

$$e_{a,i,j,k}e^{a,l,m,n} = d_i^l d_j^m d_k^n + d_i^m d_j^n d_k^l + d_i^n d_j^l d_k^m$$

$$- d_i^l d_j^n d_k^m - d_i^m d_j^l d_k^n - d_i^n d_j^m d_k^l$$

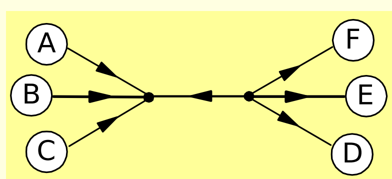


Determinant of Product of Matrices

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ & & & \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix} \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ & & \\ \cdot & \cdot & \cdot \end{bmatrix}$$

$$\begin{bmatrix} - & \mathbf{A} & - \\ - & \mathbf{B} & - \\ - & \mathbf{C} & - \end{bmatrix} \begin{bmatrix} \vdots & \vdots & \vdots \\ \mathbf{D} & \mathbf{E} & \mathbf{F} \\ \vdots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} \mathbf{A}\cdot\mathbf{D} & \mathbf{A}\cdot\mathbf{E} & \mathbf{A}\cdot\mathbf{F} \\ \mathbf{B}\cdot\mathbf{D} & \mathbf{B}\cdot\mathbf{E} & \mathbf{B}\cdot\mathbf{F} \\ \mathbf{C}\cdot\mathbf{D} & \mathbf{C}\cdot\mathbf{E} & \mathbf{C}\cdot\mathbf{F} \end{bmatrix} = \mathbf{M}$$

$$\begin{bmatrix} - & cross(\mathbf{D},\mathbf{E},\mathbf{F}) & - \end{bmatrix} \begin{bmatrix} \vdots \\ cross(\mathbf{A},\mathbf{B},\mathbf{C}) \\ \vdots \end{bmatrix} = \det \mathbf{M}$$



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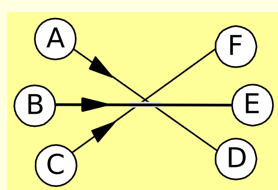

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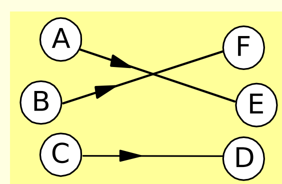

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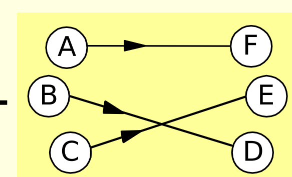

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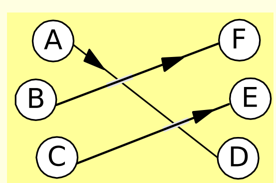

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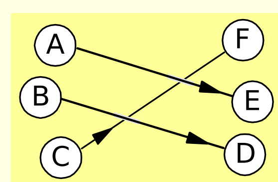

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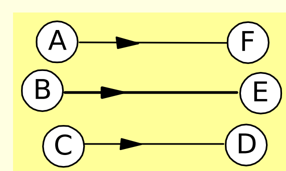
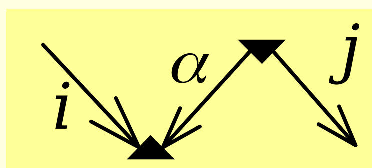


2D(1DH) Epsilon Delta

$$\epsilon_{\alpha,i}\epsilon^{\alpha,j} = \delta_i^j$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



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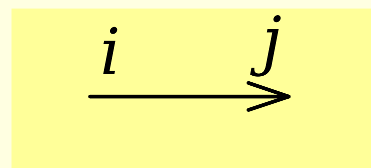

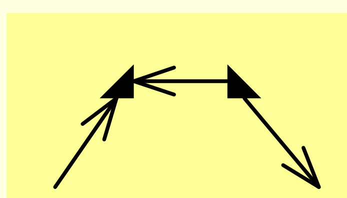

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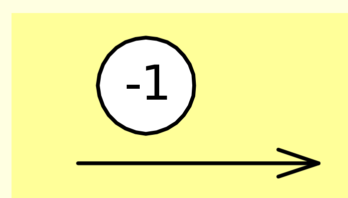
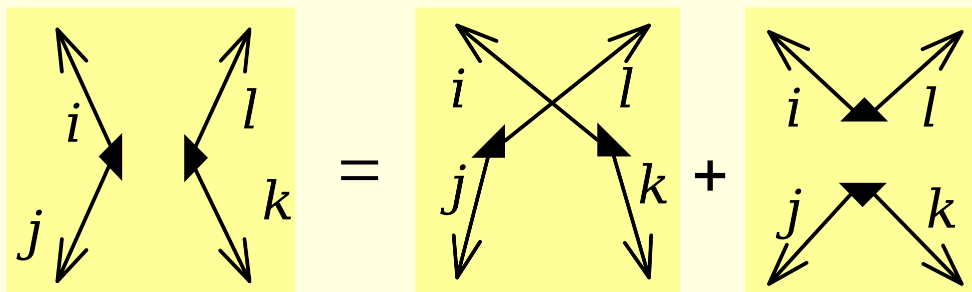
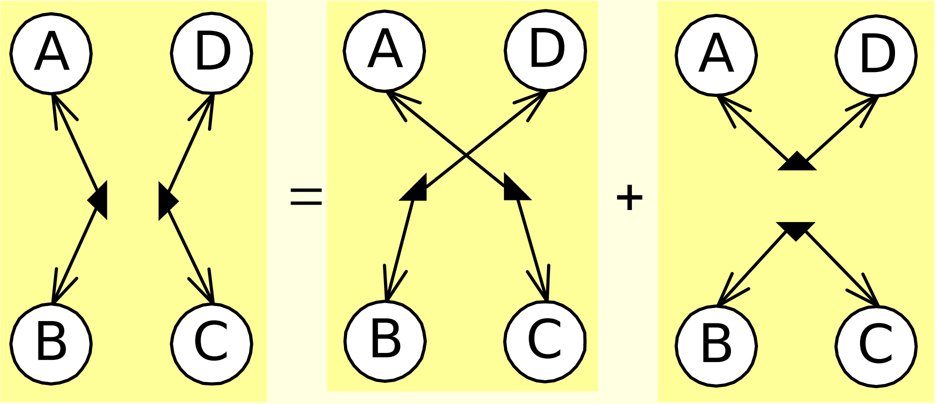


Another Epsilon Delta

$$e_{i,j}e_{k,l} = e_{i,k}e_{j,l} - e_{i,l}e_{j,k}$$



Another Interpretation



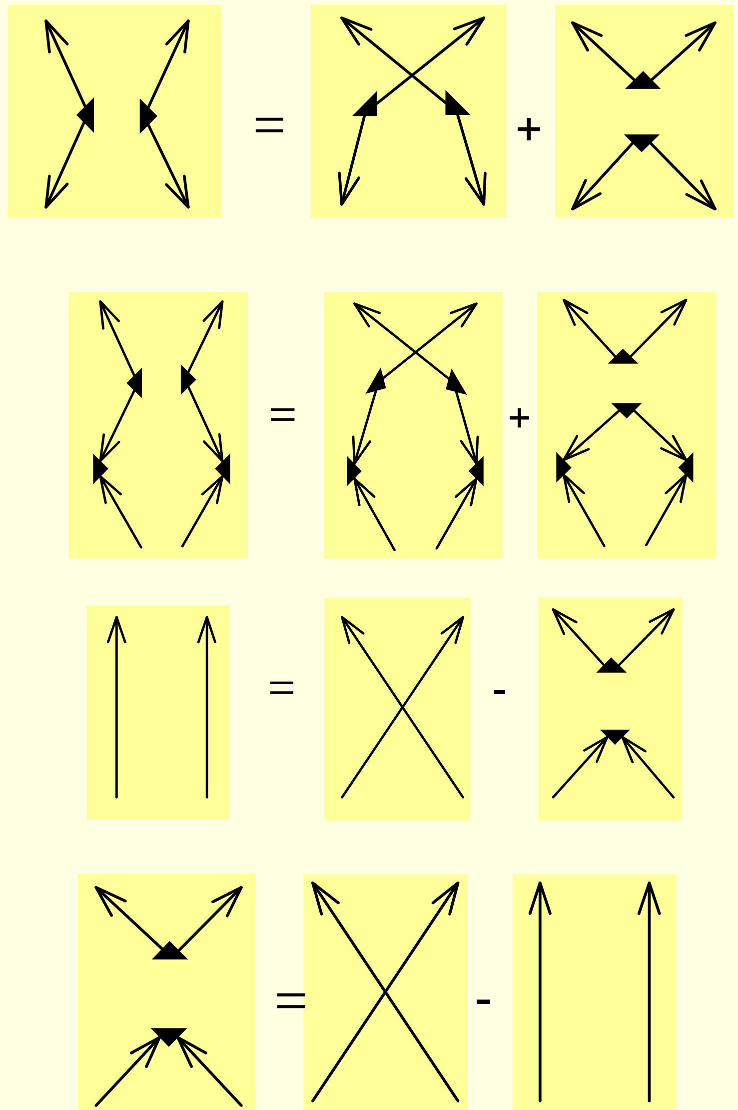
$$(A-B)(C-D)=(A-C)(B-D)+(B-C)(D-A)$$

$$AC-AD-BC+BD+(AB-AB)+(CD-CD)$$

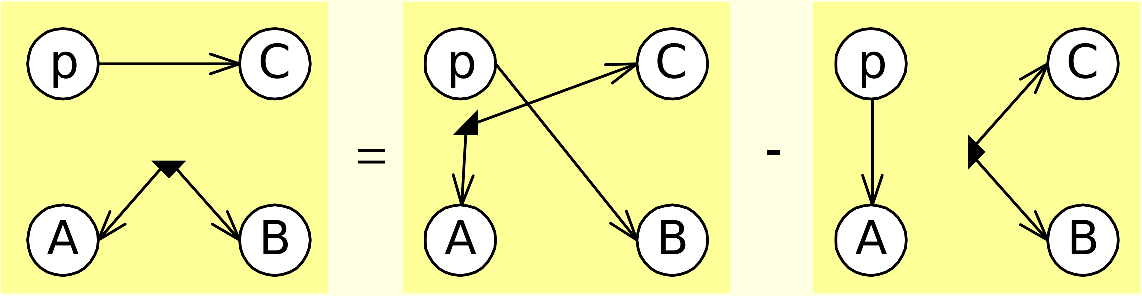
$$=AB-BC-AD+CD$$

$$+BD-CD-AB+AC$$

Yet Another Epsilon Delta

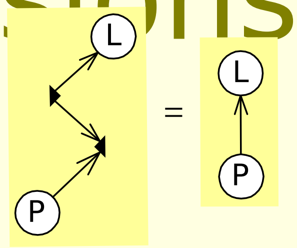


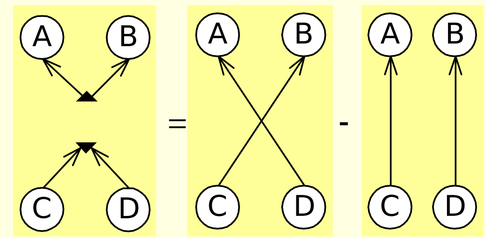
Yet Yet Another



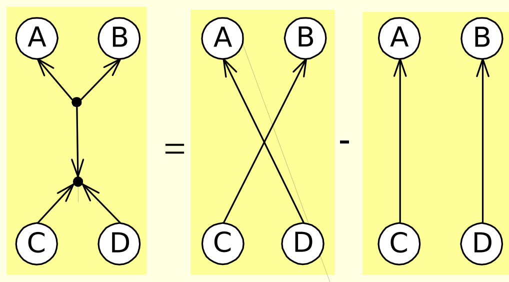
EpsDel in all Three Dimensions

2D
(1DH)

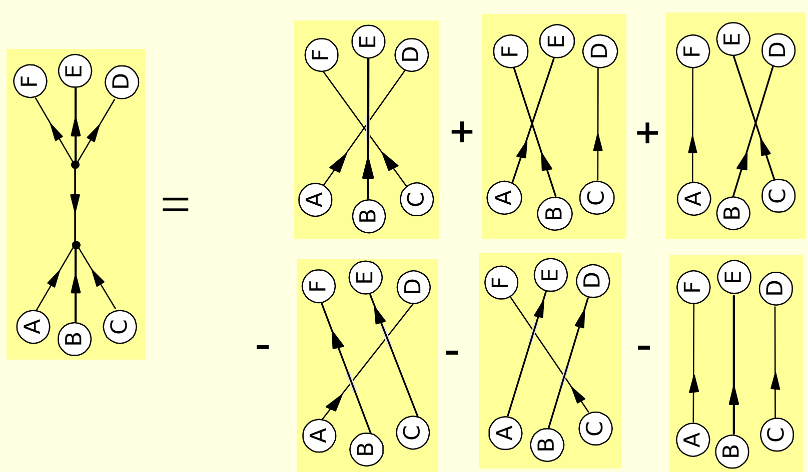




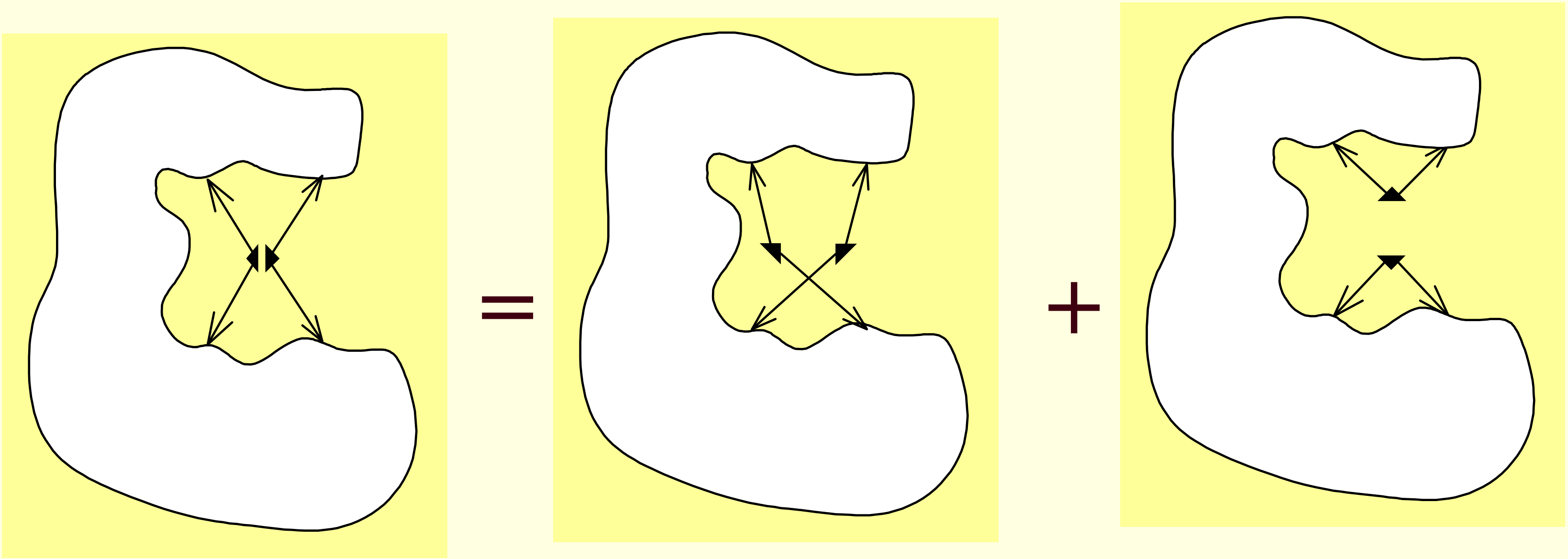
3D
(2DH)



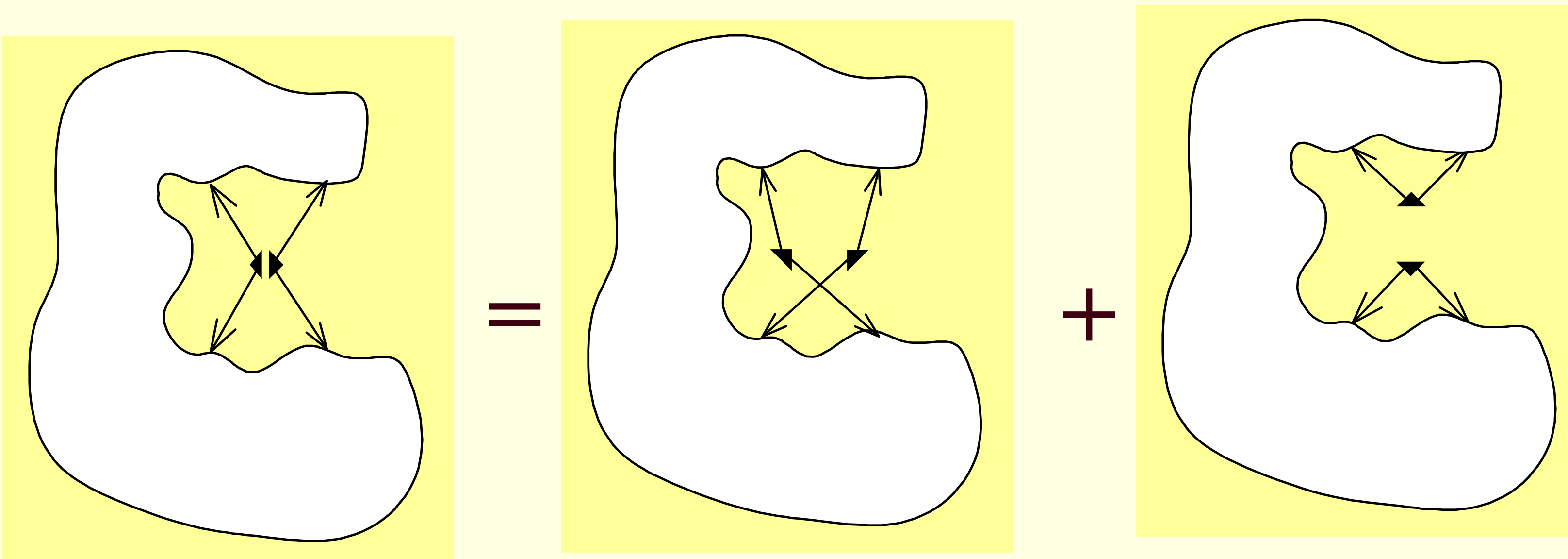
4D
(3DH)



Usage Strategy

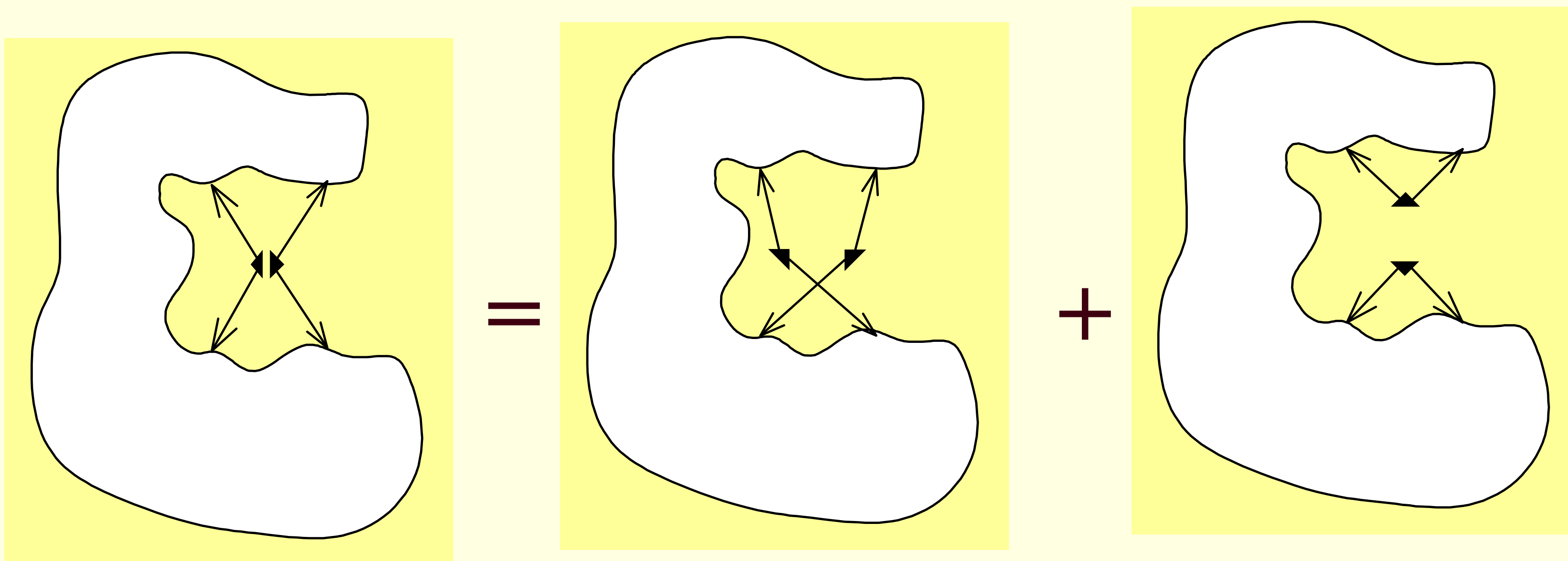


Usage Strategy



$$D_1 = 0 + D_1$$

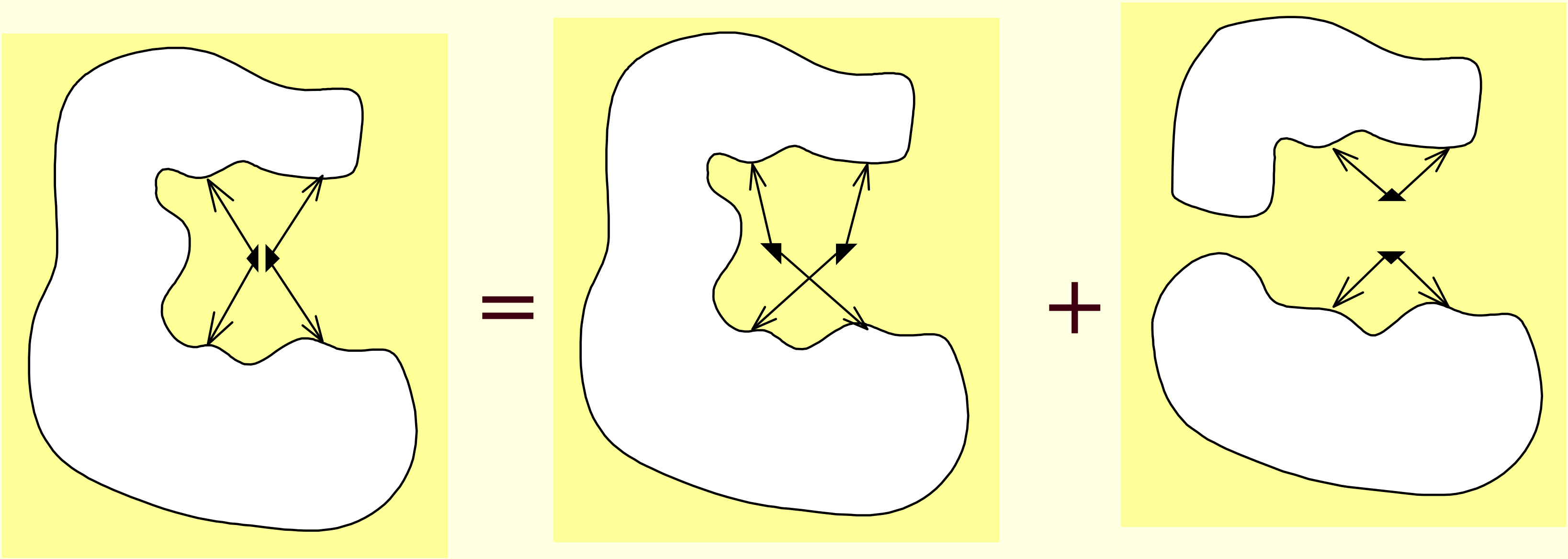
Usage Strategy



$$D_1 = - D_1 + D_2$$

$$D_1 = \tfrac{1}{2} D_2$$

Usage Strategy

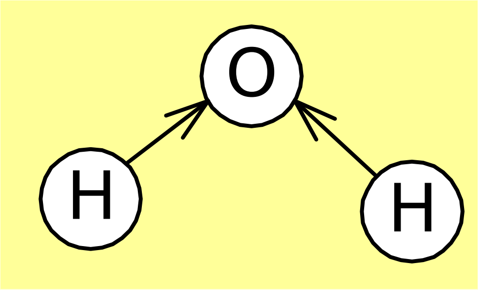


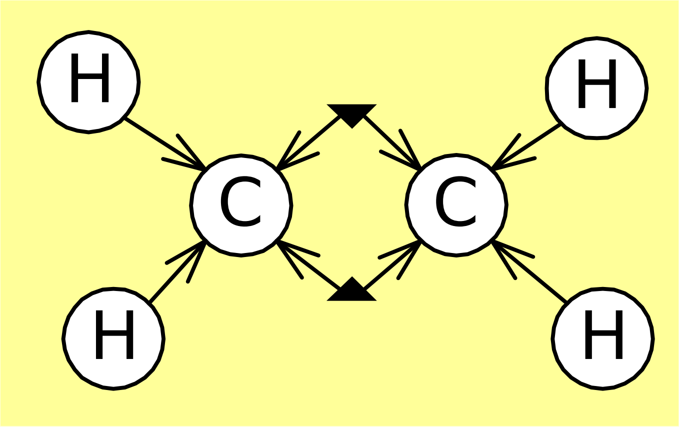
$$D_1 = 0 + d_2 d_3$$

Substitution

Sub-Atomic Particles

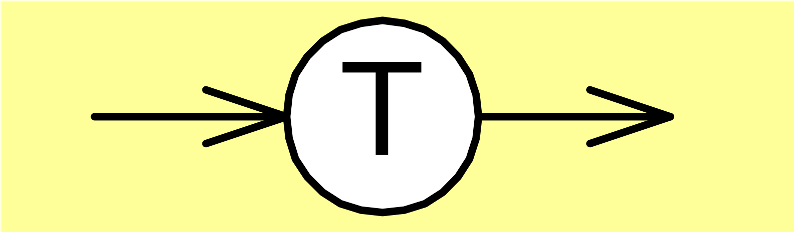
Molecules





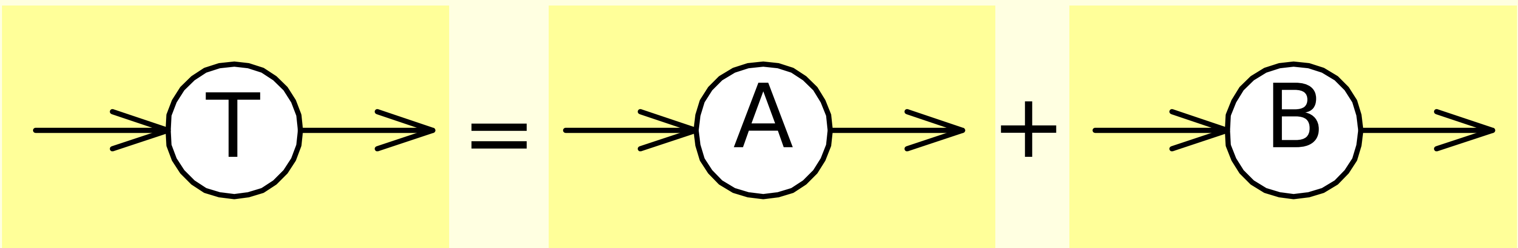
SubAtomic Physics

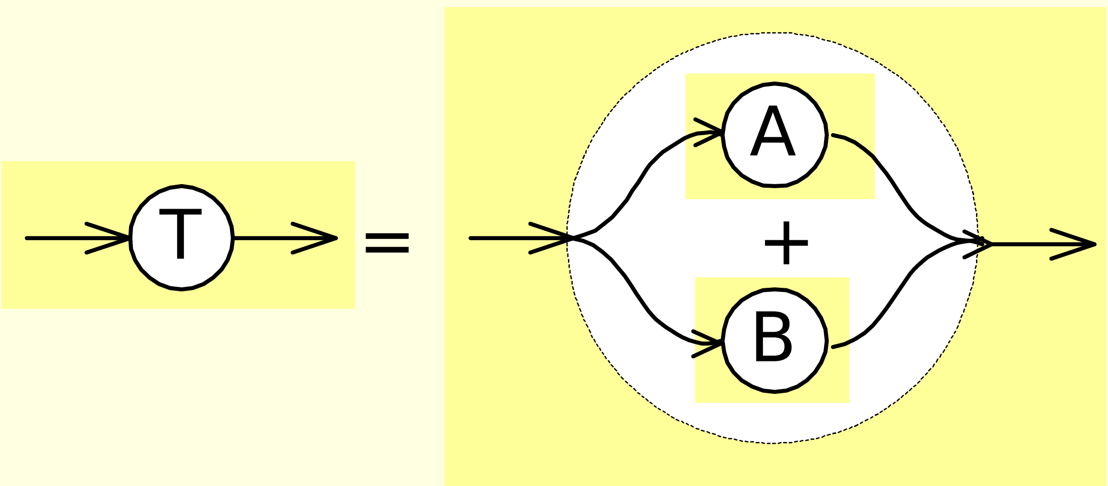
What's Inside?



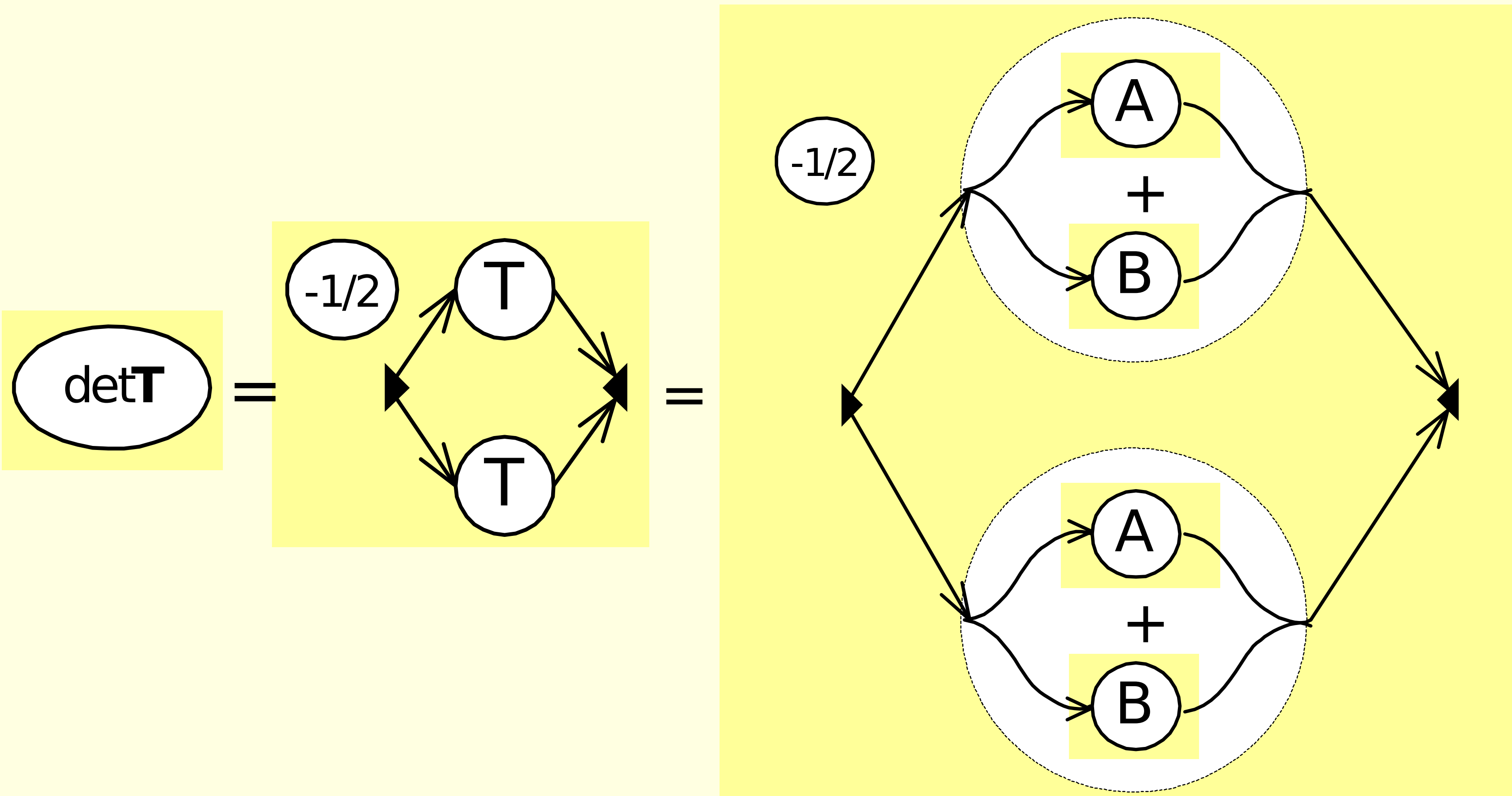
Sum of Matrices

$$\mathbf{T} = \mathbf{A} + \mathbf{B}$$

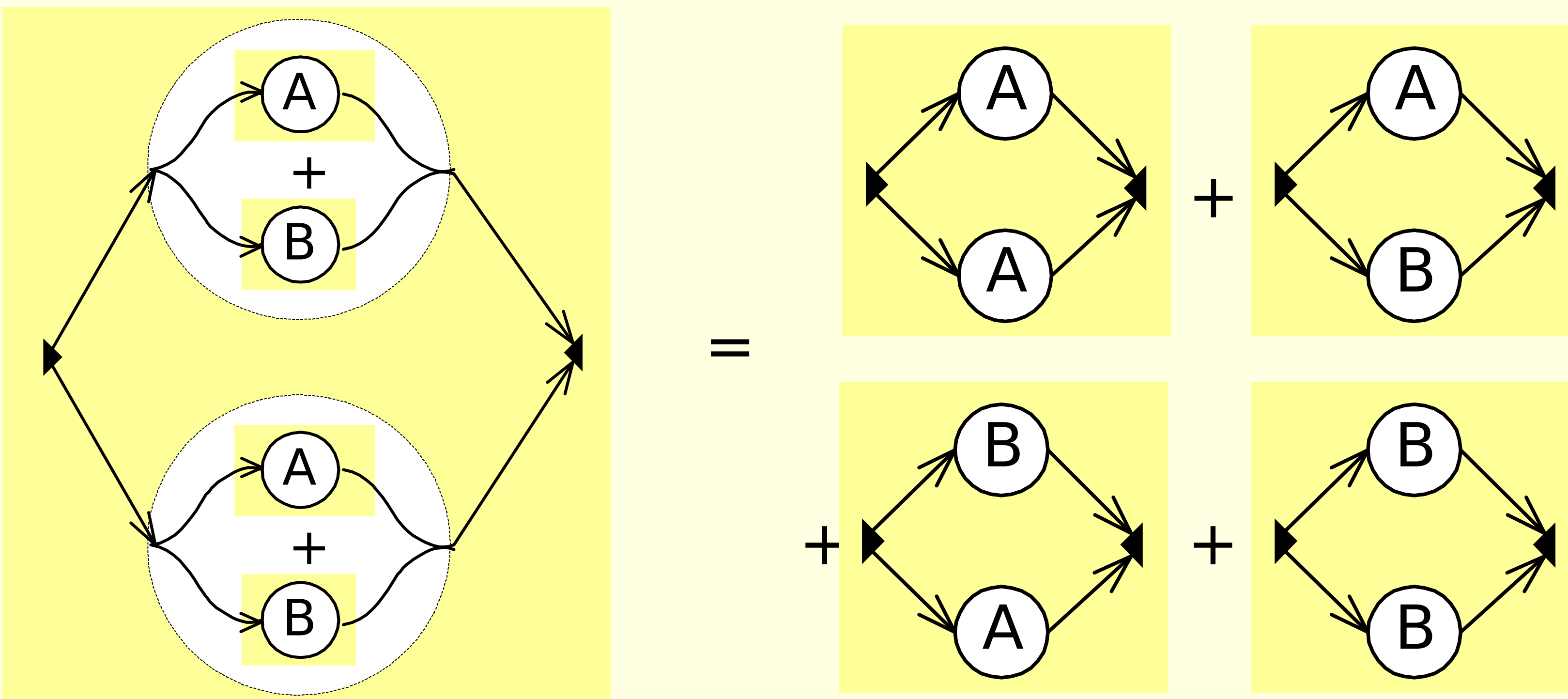




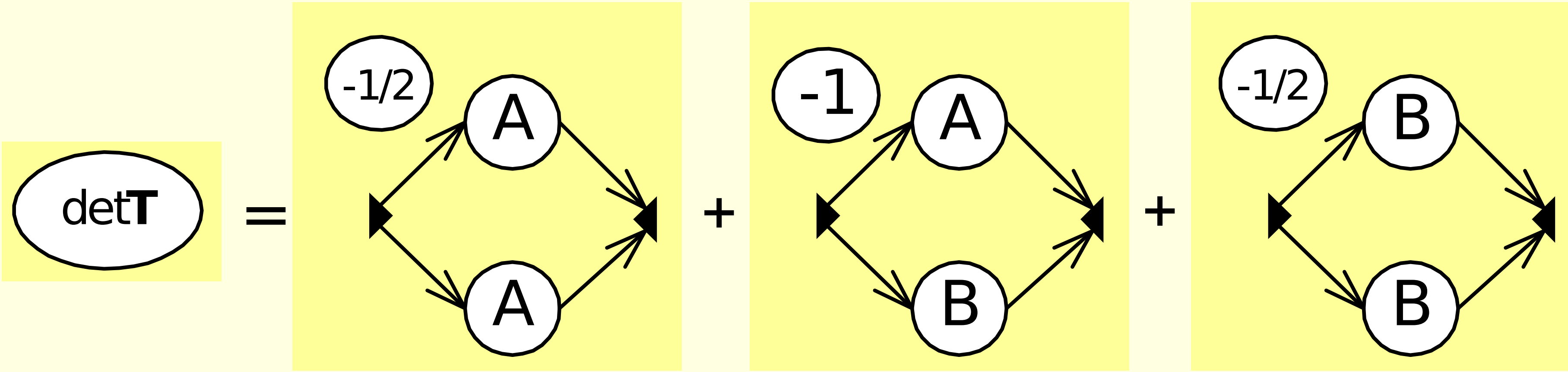
Determinant of T



Determinant of T



Determinant of T



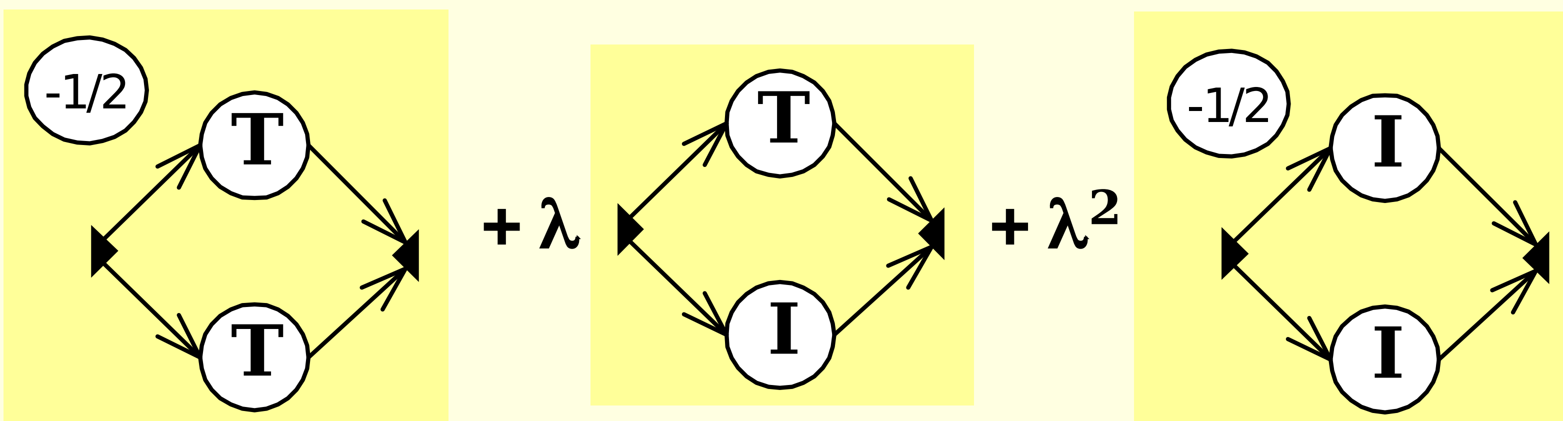
$$\det(\mathbf{A}+\mathbf{B}) = \det\mathbf{A} + fcn(\mathbf{A},\mathbf{B}) + \det\mathbf{B}$$

Eigenvectors/Eigenvalues

$$\mathbf{TL} = / \ \mathbf{L}$$

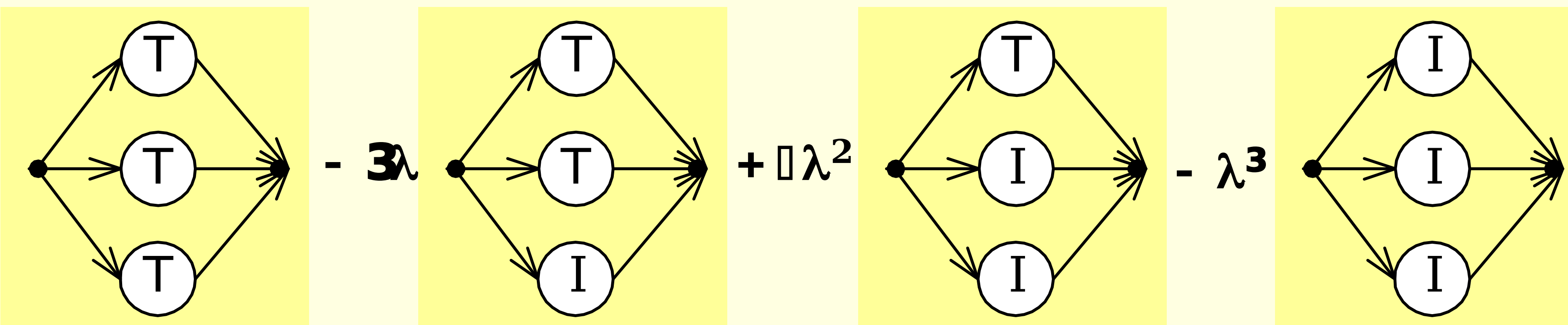
Characteristic Equation 2D(1DH)

$$\det\left(\mathbf{T} - / \ \mathbf{I}\right) = 0$$



Characteristic Equation 3D(2DH)

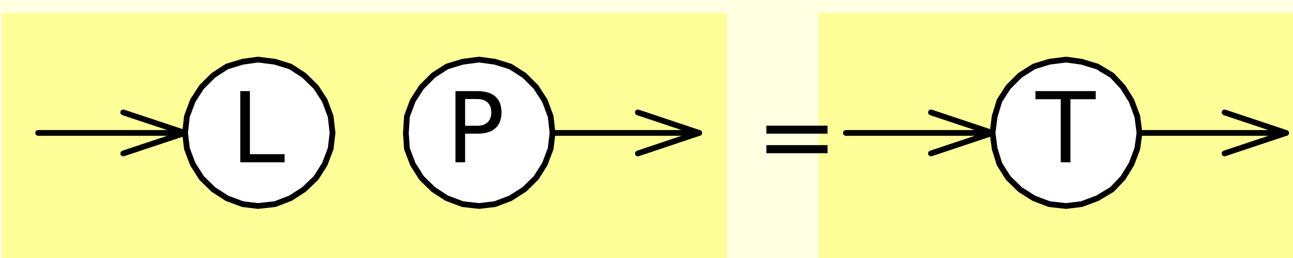
$$\det\left(\mathbf{T} - \lambda \mathbf{I}\right) = 0$$



Outer Product

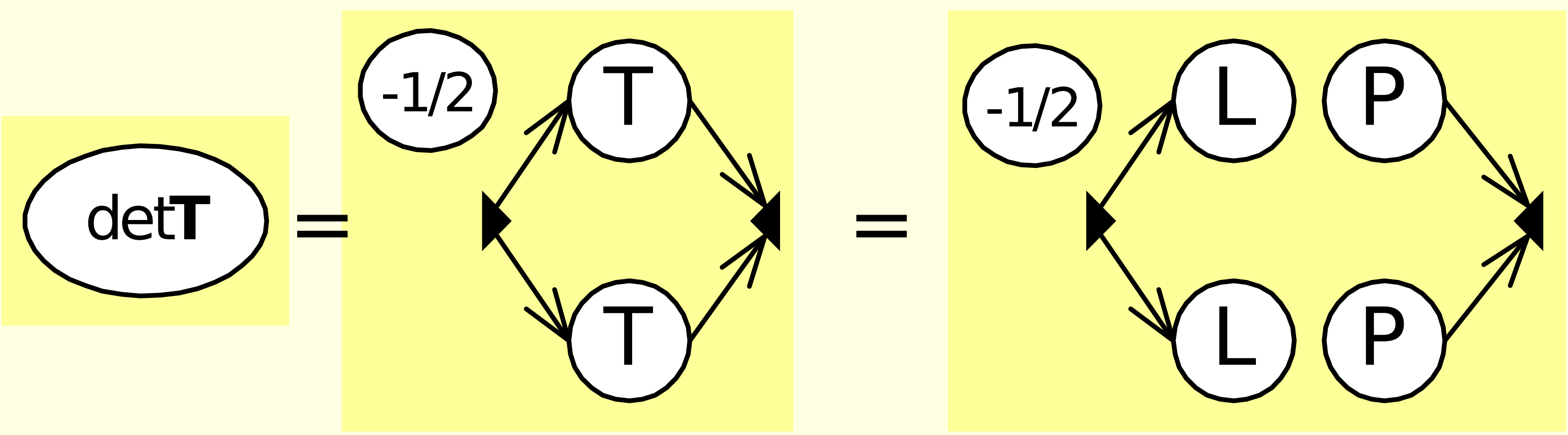
$$\begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} x & w \end{bmatrix} = \begin{bmatrix} ax & aw \\ bx & bw \end{bmatrix}$$

$$\mathbf{L}\,\mathbf{P} = \mathbf{T}$$



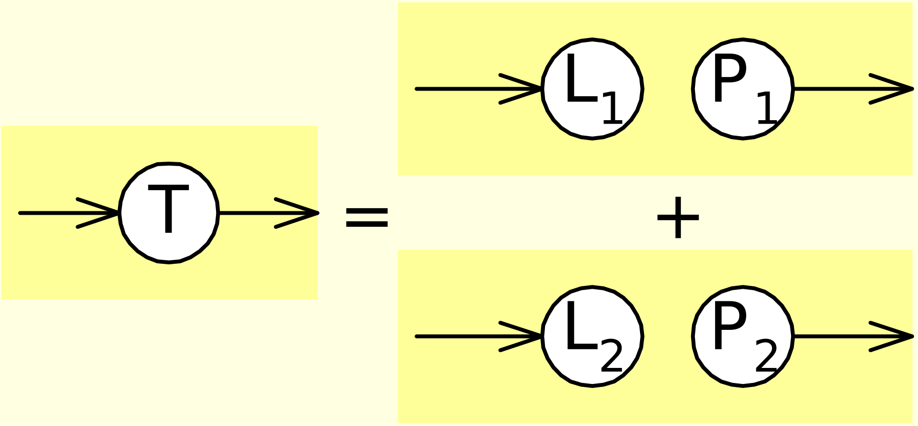
Outer Product is Singular

$$\det\begin{bmatrix} ax & aw \\ bx & bw \end{bmatrix} = axbw - bxaw = 0$$

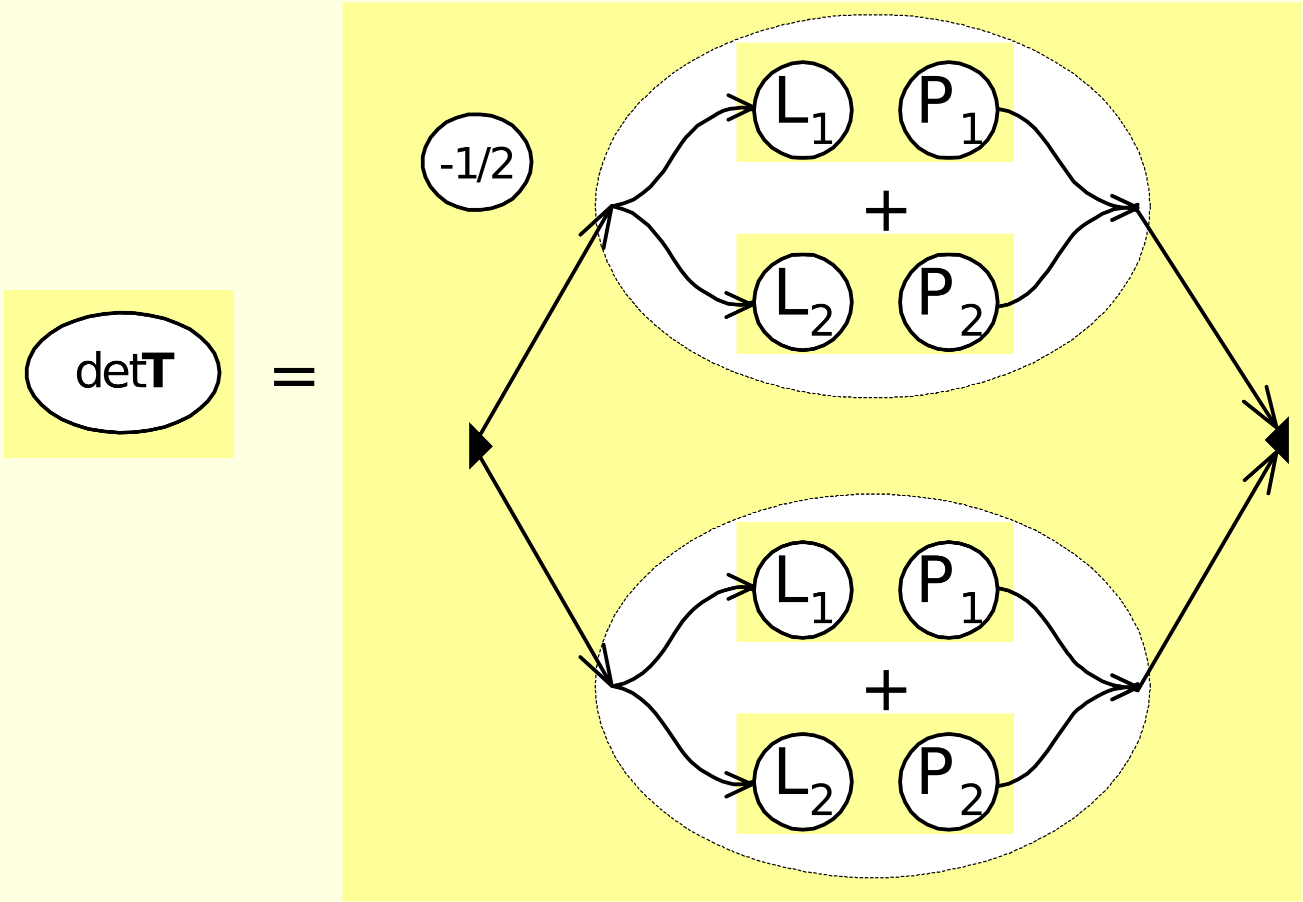


Sum of Outer Products

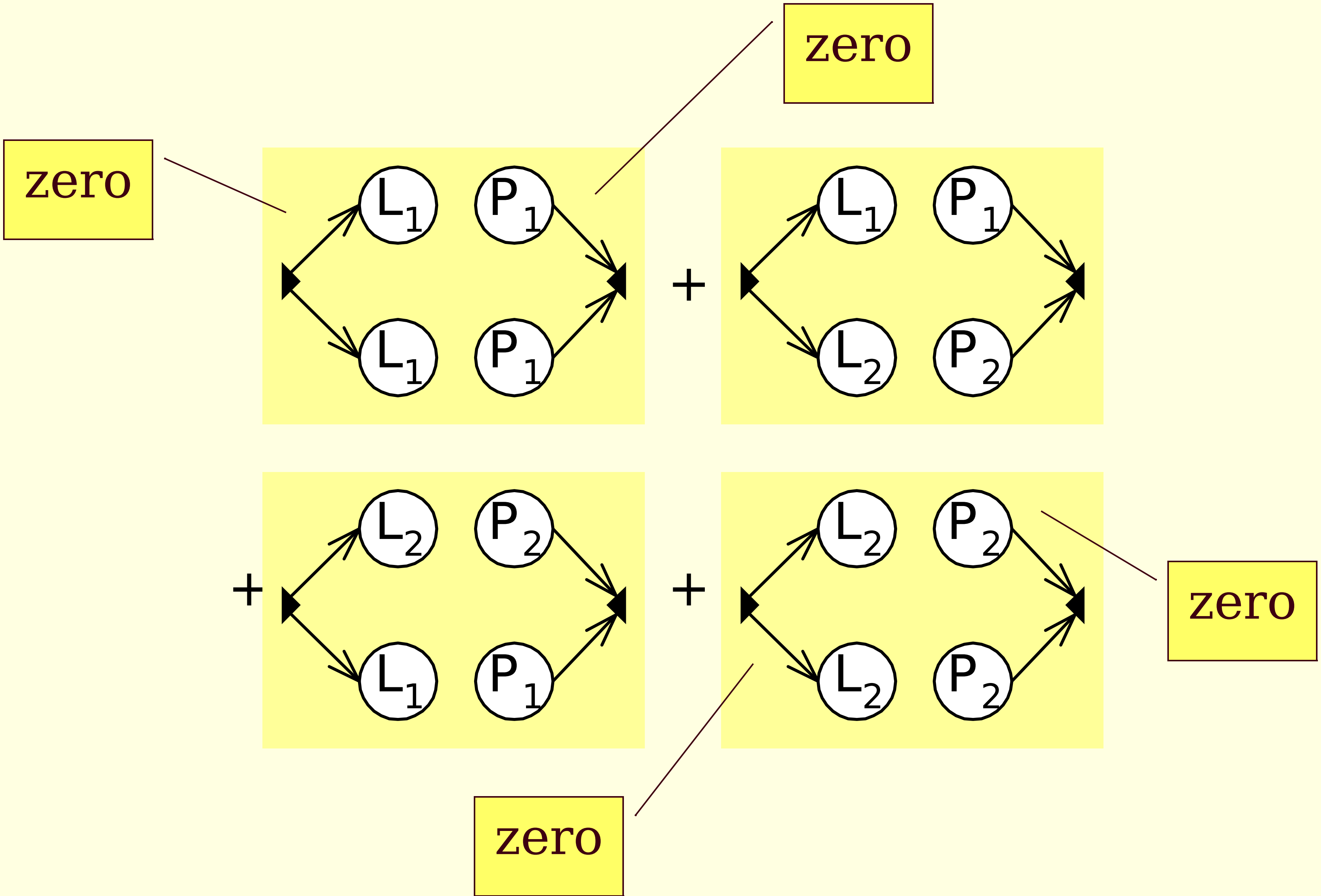
$$\mathbf{T} = \mathbf{L}_1\mathbf{P}_1 + \mathbf{L}_2\mathbf{P}_2$$



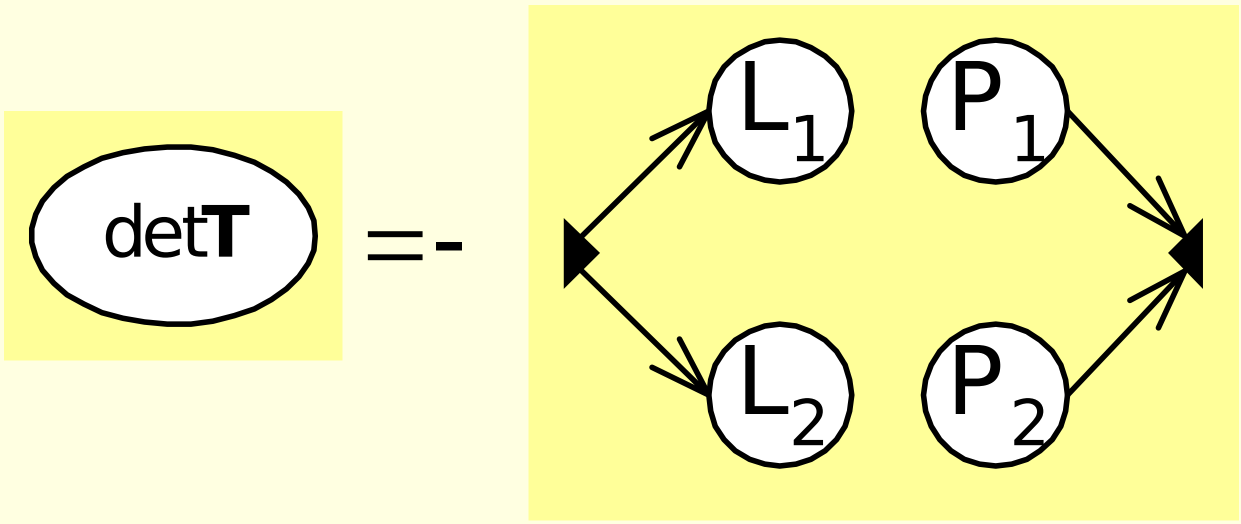
Determinant of Sum of Outer Products



Determinant of Sum of Outer Products

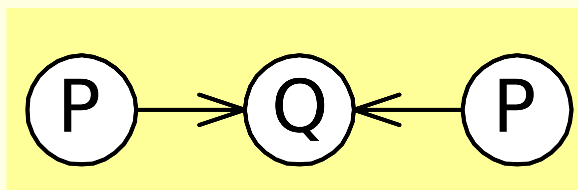


Determinant of Sum of Outer Products



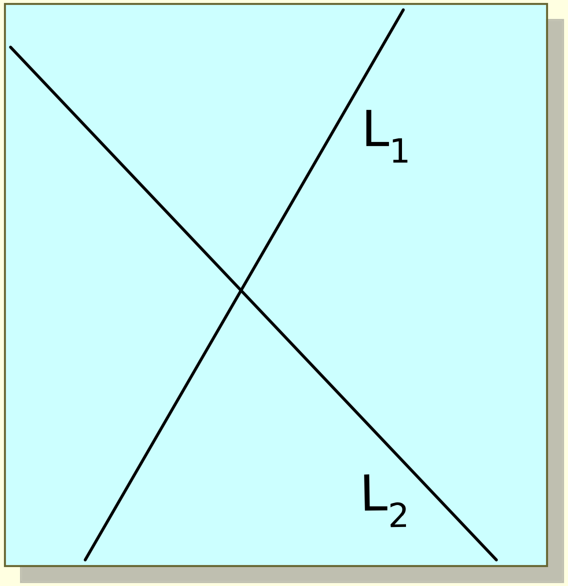
Symmetric Tensors

$$\begin{bmatrix} x & y & w \end{bmatrix} \begin{bmatrix} A & B & D \\ B & C & E \\ D & E & F \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = \mathbf{PQP}^T = 0$$



$= 0$

Factorable Quadratic Tensor



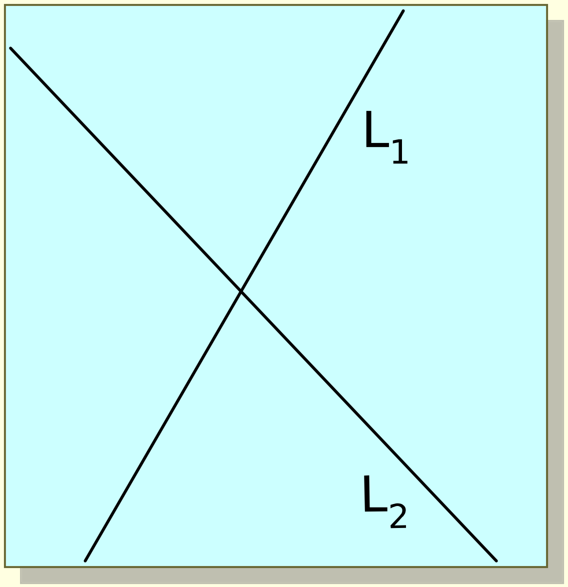
$$\left(\mathbf{P}\mathbf{L}_1\right)\left(\mathbf{P}\mathbf{L}_2\right)=0$$

$$=\begin{bmatrix} x & y & w \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}\begin{bmatrix} p & q & r \end{bmatrix}\begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

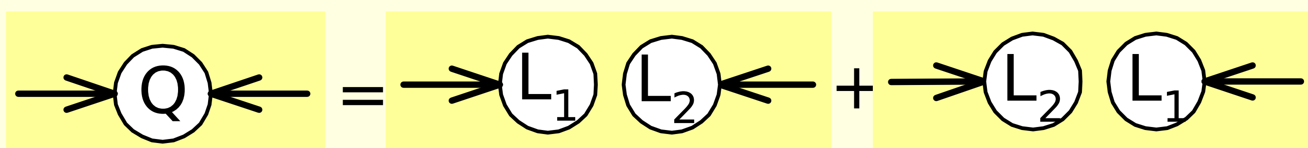
$$=\mathbf{P}\left(\mathbf{L}_1\mathbf{L}_2{}^T\right)\mathbf{P}^T$$

$$\mathbf{Q}\overset{?}{=}\mathbf{L}_1\mathbf{L}_2{}^T=\begin{bmatrix} a \\ b \\ c \end{bmatrix}\begin{bmatrix} p & q & r \end{bmatrix}=\begin{bmatrix} ap & aq & ar \\ bp & bq & br \\ cp & cq & cr \end{bmatrix}$$

Factorable Quadratic Tensor

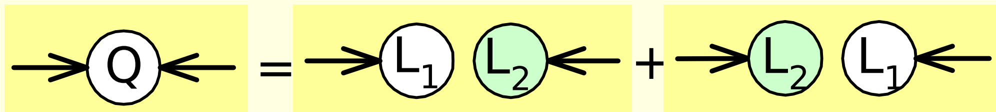


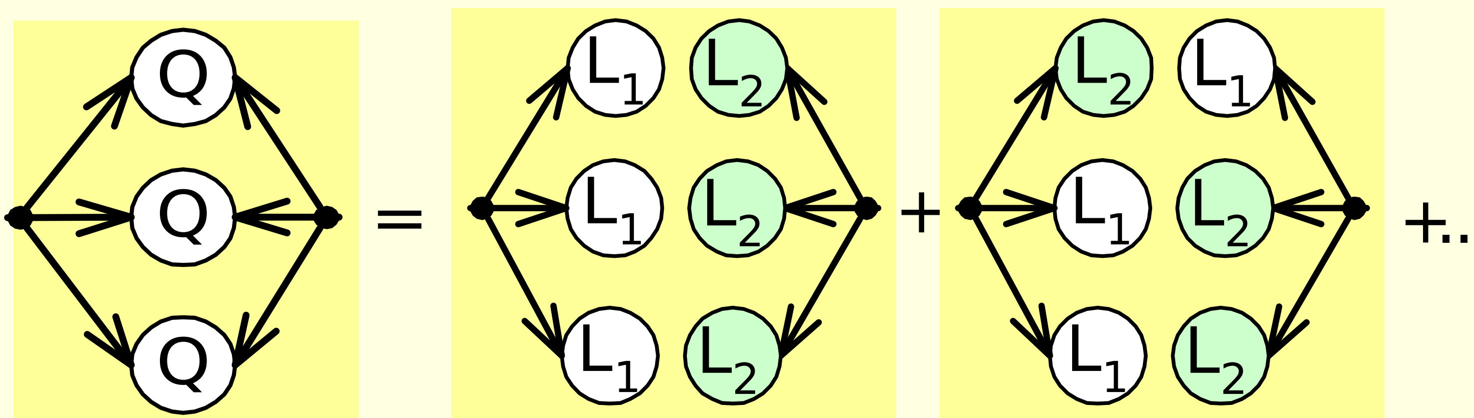
$$\mathbf{Q} = \mathbf{L}_1 \mathbf{L}_2^T + \mathbf{L}_2 \mathbf{L}_1^T$$



$$\mathbf{PQP}^T = \mathbf{PL}_1 \mathbf{L}_2^T \mathbf{P}^T + \mathbf{PL}_2 \mathbf{L}_1^T \mathbf{P}^T = 2\left(\mathbf{PL}_1\right)\left(\mathbf{PL}_2\right)$$

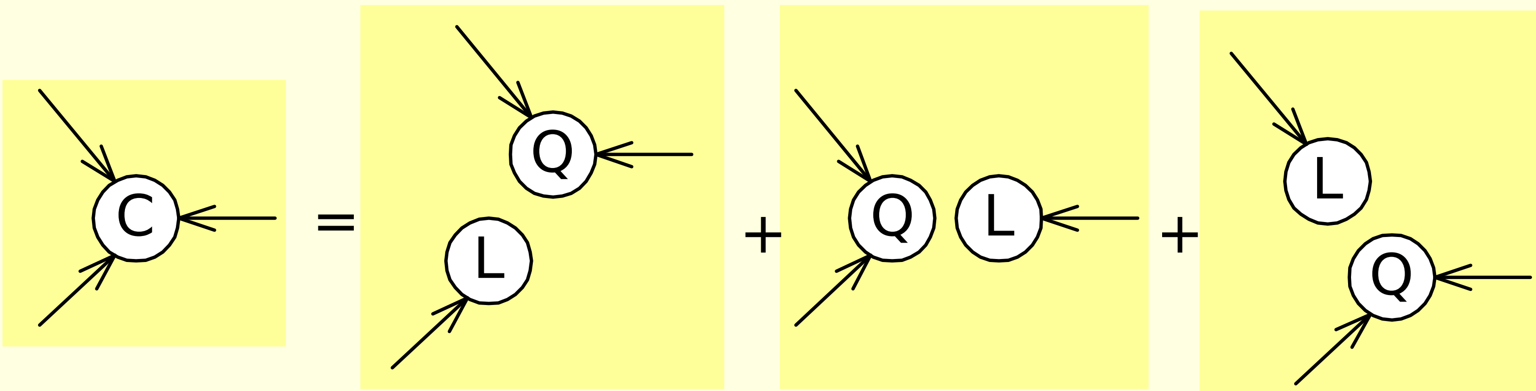
Determinant of Factorable Quadratic



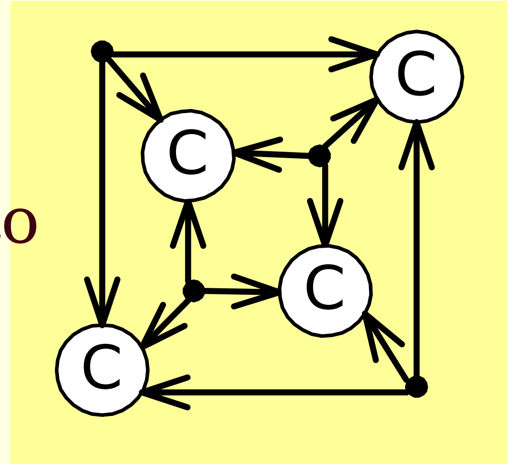


$$= 0$$

Factorable Cubic Tensor



Substitute Into



After The Break

- Polynomial Roots and Discriminants
- Polynomial Resultants and Generalizations
- Quadratic and Curves and Theorem of Pascal
- Properties of Cubic Curves